Notes for Introduction to Real Analysis

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1 Preliminaries

1.1 Notation

In this article, $\mathbb{N} = \{1, 2, 3, \cdots\}$ will **not** contain 0. We will denote with $\mathbb{Z}_{\geq 0}$ the set of natural numbers, with 0 included.

We will assume everything about $\mathbb{Q} := \mathbb{Z} \times \mathbb{N}$, including, but not limited, to the dense property, the unboundedness, the field axioms, the inequalities, etc.

Unless specified otherwise, all variables are assumed to be real.

1.2 Sets

We will use the typical set notations. For example:

- $\{1, 2, \cdots\}$ refers to the set of natural numbers, $\mathbb{N}$;
- $\{p/q \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$ refers to the set of rational numbers, $\mathbb{Q}$. We also use ":" instead of "|" sometimes to avoid ambiguity;
- $\{x \in \mathbb{R} \mid \exists (p, q) \in \mathbb{Z} \times \mathbb{N}, x = 2p/q\}$ refers to the same set (why?), where $\times$ denotes the Cartesian product of sets. We also use : instead of | sometimes to avoid ambiguity.

We use the symbol $\varnothing$ to denote the empty set. We use $|\cdot|$ to denote the cardinality of a set. We use the symbol 2^S to denote the power set of S, or the set of all subsets of S. We use B^A to denote the set of all functions from A to B.

To justify the use of cardinality, we state the following definition:

Definition 1.1. Let A, B be sets. A and B are said to have the same cardinality iff there exists a bijective function $A \to B$; that is, $\exists f \in B^A, (\forall y \in B, \exists x \in A, f(x) = y) \wedge (\forall (x_1, x_2) \in A \times A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

We say that $|A| \leq |B|$ if there exists an injection from A to B. We say that $|A| = |B|$ if A and B have the same cardinality. We say that $|A| < |B|$ if $|A| \leq |B|$ and $\neg(|A| = |B|)$.

We state the following without proof.

Theorem 1.2 (Cantor-Bernstein-Schröder theorem). *Let A, B be sets such that $|A| \leq |B|$ and $|B| \leq |A|$. Then, $|A| = |B|$.*

If you're interested, click here to read about a proof!

Definition 1.3. Suppose A is a set. If $A = \varnothing$, then we write $|A| = 0$. If $|A| = |\{1, 2, \cdots, n\}|$ for some $n \in \mathbb{N}$, then we write $|A| = n$. If either condition holds, then we say that A is finite. If A is not finite, we say that A is infinite.

If $|A| = |\mathbb{N}|$, then we say that A is countably infinite and write $|A| = \aleph_0$.[1]

[1] The first Hebrew letter, ℵ, is pronounced as /ˈɑːlef/, similar to AH-lef ("lef" as in "left") in English.

Three big things we state without proof:

- Countably infinite sets are infinite;
- An infinite set has a countably infinite subset;
- Suppose A, B are sets. If $A \subset B$ and B is finite, then A is finite. The contrapositive is if $A \subset B$ and A is infinite, then B is infinite.

An immediate consequence of the last statement is that $|A| < |\mathbb{N}|$ implies that A is finite for any set A.

2 The Set of Real Numbers

We should start by claiming:

Theorem 2.1. *The set of real numbers $\mathbb{R}$ exists and is a complete ordered field.*[2]

If you are interested in a proof, see Appendix A.

Definition 2.2. A subset of real numbers $S \subseteq \mathbb{R}$ is said to be bounded from above iff $\exists M \in \mathbb{R}, \forall x \in S, x \leq M$; such M is said to be an upper bound of S. Similarly, a subset of real numbers $S \subseteq \mathbb{R}$ is said to be bounded from below iff $\exists M \in \mathbb{R}, \forall x \in S, x \geq M$; such M is said to be a lower bound of S. A subset $S \subseteq \mathbb{R}$ is bounded iff it is both bounded from above and bounded from below.

An upper/lower bound is said to be strict iff the inequality in the definition can be replaced with a strict inequality.

Definition 2.3. Let $S \subseteq \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$. We say that b_1 is a least upper bound, or supremum, of S iff b_1 is an upper bound and b_1 is no greater than any upper bound, or formally,

$$\forall b' \in \mathbb{R}, (\forall x \in S, x \leq b') \Rightarrow b' \geq b_1.$$

Similarly, b_2 is said to be a greatest lower bound, or infimum, of S off b_2 is a lower bound of S and b_2 is no less than any lower bound, or formally,

$$\forall b' \in \mathbb{R}, (\forall x \in S, x \geq b') \Rightarrow b' \leq b_2.$$

Proposition 2.4. Let $S \subseteq \mathbb{R}$ be non-empty. If b is a least upper bound of S, then b is unique.

Proof. Let $S \subset \mathbb{R}$ be non-empty. Let b be a least upper bound of S. Then, b is no greater than any *other* upper bound b', or $b' > b$. So b' cannot be the *least* upper bound, and the proof is completed. □

As noted above, we will take for granted the least upper bound property of real numbers, which can be derived from, e.g., Dedekind cuts, as demonstrated in Theorem A.11.

Theorem 2.5. *Let $S \subset \mathbb{R}$. If S is bounded from above, then S has a least upper bound.*

We now present another equivalent characterization of least upper bounds.

Proposition 2.6. Let $S \subset \mathbb{R}$ be non-empty and bounded from above. For any upper bound b of S, b is the least upper bound of S if and only if $\forall \epsilon > 0, \exists x \in S, b - x < \epsilon$.

Proof. We first prove the "if" direction. Suppose b is an upper bound of S and $\forall \epsilon > 0, \exists x \in S, b - x < \epsilon$. Suppose $b' < b$ is also an upper bound of S. Let $\epsilon = b - b'$. Then, there exists some $x \in S$ such that $b - x < b - b'$. That is, $b' < x$ for some $x \in S$, so b' is not an upper bound of S, which is a contradiction.

[2] The word "complete" is in the sense that $\mathbb{R}$ has the least upper bound property.

We now show the "only if" direction. Suppose b is the least upper bound of S. Then, any $b' < b$ is not an upper bound; that is, $\forall b' < b, \exists x \in S, x > b'$. For any $\epsilon > 0$, let $b' = b - \epsilon$. Since there exists some $x \in S$ such that $x > b'$, we have $b - \epsilon < x$. The proof is completed □

We now have all the tools we need to construct powers of the form x^n with integer values of n. Before our construction, we will give two important properties that we assert powers to satisfy:

Property 2.7. Suppose $x > 0$ and $a, b \in \mathbb{R}$. Then, $(x^a)^b = x^{ab}$.

Property 2.8. Suppose $x > 0$ and $a, b \in \mathbb{R}$. Then, $x^a \cdot x^b = x^{a+b}$.

Definition 2.9. Suppose x is real and $n \in \mathbb{Z}$. We define $x^n = 1$ if $n = 0$.[3] Otherwise, we define recursively

$$x^{n+1} = n \cdot x^n \quad (n \in \mathbb{Z}).$$

We will, however, begin to prove the binomial theorem.

Theorem 2.10. *Suppose $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then,*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^{n-i} \cdot y^i.$$

Proof. We perform induction on n.

Basic step: When $n = 1$, we have $(x+y)^n = x + y$ and $\sum_{i=0}^{n} \binom{n}{i} \cdot x^{n-i} \cdot y^i = x + y$.

Inductive step: Suppose $(x+y)^n = \sum_{n=0}^{i} \binom{n}{i} \cdot x^{n-i} \cdot y^i$ for $n = k$ ($k \in \mathbb{N}$). Then,

$$\begin{aligned}
(x+y)^{k+1} &= (x+y) \cdot (x+y)^k \\
&= (x+y) \cdot \sum_{i=0}^{k} \binom{k}{i} \cdot x^{k-i} y^i \\
&= x \cdot \sum_{i=0}^{k} \binom{k}{i} \cdot x^{k-i} y^i + y \cdot \sum_{i=0}^{k} \binom{k}{i} \cdot x^{k-i} y^i \\
&= \sum_{i=0}^{k} \binom{k}{i} \cdot x^{k+1-i} y^i + \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^{i+1} \\
&= \sum_{i=0}^{k+1} \binom{k}{i} \cdot x^{k+1-i} y^i + \sum_{i=0}^{k+1} \binom{k}{i-1} x^{k+1-i} y^i \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} \cdot x^{k+1-i} y^i.
\end{aligned}$$

That is, the assumption is also true for $n = k + 1$. Therefore, by mathematical induction, the proof is finished. □

2.1 Useful Facts about Real Numbers

A very commonly used fact about real numbers is that every non-empty, finite set

3 Sequences & Series

3.1 Sequences and Limits

Our first big topic will be sequences and various ways to talk about how they behave at infinity. A sequence has a discrete domain, so a lot of counting-related stuff can be useful here, I guess.

[3] By this definition, $0^0 = 1$.

Definition 3.1. A sequence $(x_n)_{n=a}^{b}$ ($a, b \in \mathbb{N} \cup \{-\infty, +\infty\}$ and $a < b$) of real numbers is a function from the subset of integers between a and b (inclusive) to $\mathbb{R}$.

As you probably expected, we'll be playing around with limits a lot. But before all that, we need to establish the concept of boundedness.

Definition 3.2. A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to be bounded from above iff there exists some $M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \leq M$. Similarly, (x_n) is said to be bounded from below if there exists some $M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, x_n \geq M$. We say that (x_n) is bounded iff it is bounded both from above and from below.

Now we can talk about the concept of Cauchy sequences —sequences whose terms eventually get arbitrarily close to one another. We'll see later that a Cauchy sequence *is* a convergent sequence in $\mathbb{R}$. The fact that every Cauchy sequence in $\mathbb{R}$ converges to some real number $x \in \mathbb{R}$ is called the *completeness* of $\mathbb{R}$ as a metric space. This is equivalent to, e.g., the least upper bound property (but I'll be lazy and only prove the direction from the latter to the former).

Definition 3.3. A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to be a Cauchy sequence iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \in \mathbb{N}, \min\{m, n\} > N \Rightarrow |x_m - x_n| < \epsilon.$$

Now let's define what it means for a sequence to converge to some number.

Definition 3.4. A sequence $(x_n)_{n=1}^{\infty}$ is said to converge to $x \in \mathbb{R}$, denoted as $\lim_{n\to\infty} x_n = \mathbb{R}$, iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n > N \Rightarrow |x_n - x| < \epsilon.$$

A sequence $(x_n)_{n=1}^{\infty}$ is said to be convergent iff there exists some $x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$.

It would be really nice to prove the obvious statement that a sequence is convergent if and only if it is a Cauchy sequence. But we don't have the entire toolbox yet. We will, however, state the "only if" direction.

Proposition 3.5. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.

Proof. Suppose $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$. Let $\epsilon > 0$. Then, fix $N \in \mathbb{N}$ such that $\forall n > N, |x_n - x| < \epsilon/2$. Choose arbitrary $m, n > N$. Then,

$$\begin{aligned} |x_m - x_n| &= |x_m - x + x - x_n| \\ &\leq |x_m - x| + |x_n - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

The proof is completed. □

To use the symbol $\lim x_n$ like it's a number (which we do all the time), we need to show that the limit of a sequence, if it exists, is unique.

Proposition 3.6. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence. Then, there exists a unique real number $L \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = L$.

Proof. Suppose $(x_n)_{n=1}^{\infty}$ is convergent. Let $L_1, L_2 \in \mathbb{R}$ both be limits of (x_n). Suppose $\epsilon > 0$. Then, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{cases} \forall n > N_1, |x_n - L_1| < \epsilon/2, \\ \forall n > N_2, |x_n - L_2| < \epsilon/2. \end{cases}$$

Then, $|L_1 - L_2| \leq |x_n - L_2| + |x_n - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily, we conclude that $|L_1 - L_2| = 0$, and thus $L_1 = L_2$. The proof is completed. □

Here, we used the fact that if a number is less than ϵ for any positive ϵ, then it can't be positive. Formally,

$$\forall x \in \mathbb{R}, \forall \epsilon > 0, x < \epsilon \Rightarrow x \leq 0.$$

Prove it if you want, it would be a one-liner. This really obvious fact, though, can be helpful since we play with ϵ's a lot in limit proofs.

We can now talk about the relationship between convergence and boundedness: if a sequence converges, then it is bounded. It's kind of useless on its own, but the contrapositive is used a lot: any unbounded sequence diverges.

Proposition 3.7. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$. Then, (x_n) is bounded.

Proof. Suppose (x_n) converges to x. Then for $\epsilon = 1$, there exists some $N \in \mathbb{N}$ such that $\forall n > N, |x_n - x| < 1$. Thus, for any $n > N$, $|x_n| \leq |x_n - x + x| = |x_n - x| + |x| < 1 + |x|$. Let $M = 1 + \max\{|x_1|, |x_2|, \cdots, |x_N|, 1 + |x|\}$ (A finite subset of real numbers always has a maximum). Then, for any $n \in \mathbb{N}$, $|x_n| < M$. The proof is completed. □

Now we can begin to build towards our first big theorem: the monotone convergence theorem. Of course, we need to first define monotone sequences.

Definition 3.8. A sequence $(x_n)_{n=1}^{\infty}$ is said to be monotone increasing iff $\forall n \in \mathbb{N}, x_{n+1} \geq x_n$. Similarly, (x_n) is said to be monotone decreasing iff $\forall n \in \mathbb{N}, x_{n+1} \leq x_n$. If the inequality is strict, then the sequence is said to be strictly increasing or strictly decreasing, respectively.

Now, we can state the theorem!

Theorem 3.9 (Monotone Convergence Theorem)**.** *Suppose the sequence $(x_n)_{n=1}^{\infty}$ is monotone. Then, it is convergent if and only if it is bounded. Further, if it is bounded and monotone increasing, then*

$$\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\};$$

if it is bounded and monotone decreasing, then

$$\lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}.$$

Proof. We first prove the "if" direction. Suppose (x_n) is monotone increasing and bounded. Then, let $S = \{x_n \mid n \in \mathbb{N}\}$. Since (x_n) is bounded, S is bounded as well and thus admits the least upper bound $b \in \mathbb{R}$. Then, by Proposition 2.6, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $b - x_N < \epsilon$. Let $\epsilon > 0$ be arbitrary and fix N. Then, for any $n > N$,

$$|x_n - b| = b - x_n \leq b - x_N < \epsilon,$$

so (x_n) converges to b. If (x_n) is monotone decreasing, we apply the same argument and conclude that (x_n) converges to the greatest lower bound of S.

We now show the "only if" direction. Suppose (x_n) is monotone increasing and converges to x. If (x_n) is not bounded from above, then by Proposition 3.7, it is not convergent, which is a contradiction. Therefore, (x_n) is bounded from above. Similarly, if (x_n) is monotone decreasing and converges to x, we may apply the same argument to conclude that (x_n) is bounded.

□

Let's also talk about the k-tail of the sequence, which, in many ways, exhibit many similar behaviors to the original sequence. And we'll see later on that it's actually a special type of subsequences.

Definition 3.10. Let $(x_n)_{n=1}^{\infty}$ be a sequence. The <u>k-tail ($k \in \mathbb{Z}_{\geq 0}$) of (x_n)</u> is defined as the sequence $(x_m)_{m=k+1}^{\infty}$.

We now state the following relationship:

Proposition 3.11. Let $(x_n)_{n=1}^{\infty}$ be a sequence. The following conditions are equivalent:

- (x_n) is bounded;
- The k-tail of (x_n) is bounded for some $k \in \mathbb{Z}_{\geq 0}$;
- The k-tail of (x_n) is bounded for any $k \in \mathbb{Z}_{\geq 0}$.

This is obvious from the definition of boundedness of sequences. In fact, we can replace the word "bounded" with "convergent" throughout, and the statement is still valid.

Proposition 3.12. Let $(x_n)_{n=1}^{\infty}$ be a sequence. The following conditions are equivalent:

- (x_n) is convergent;
- The k-tail of (x_n) is convergent for some $k \in \mathbb{Z}_{\geq 0}$;
- The k-tail of (x_n) is convergent for any $k \in \mathbb{Z}_{\geq 0}$.

Clearly, the third statement implies the second. So we'll start with the latter.

Proof. Clearly, the third statement implies the second.

We now show that the second statement implies the first. Suppose for some $k \in \mathbb{Z}_{\geq 0}$, the k-tail of (x_n) converges to $x \in \mathbb{R}$. That is, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$, denoted as N_ϵ, such that $\forall n > N, |x_{n+k} - x| < \epsilon$. Suppose $\epsilon > 0$ is arbitrary. Then, for any $n > N_\epsilon + k$, let $n' = n - k > N_\epsilon$, which implies

$$|x_n - x| = |x_{n'+k} - x| < \epsilon.$$

We conclude by showing that the first statement implies the third. Suppose (x_n) converges to x. Choose any $k \in \mathbb{Z}_{\geq 0}$. For any $\epsilon > 0$, there exists some $N_\epsilon \in \mathbb{N}$ such that $\forall n > N_\epsilon, |x_n - x| < \epsilon$. Fixing n, for any $n' = n + k$, $n' > N_\epsilon$ also, so $|x_{n'} - x| < \epsilon$. The proof is completed. □

We now define what a subsequence is.

Definition 3.13. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence. A subsequence $(x_{n_i})_{i=1}^{\infty}$ of (x_n) is the composition of the function characterizing the sequence with some strictly increasing function $n\colon \mathbb{N} \to \mathbb{N}$.

There are many ways whereby sequences are related to its subsequences.

Proposition 3.14. Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence. Then, any subsequence $(x_{n_i})_{i=1}^{\infty}$ of (x_n) is also bounded.

Proof. Fix $M > 0$ such that $\forall n \in \mathbb{N}, |x_n| < M$. Then, $\forall i, n_i \in \mathbb{N}$, so $\left|x_{n_i}\right| < M$. The proof is completed. □

Of course, if a subsequence is bounded, the original sequence does not necessarily need to be. This is different from the case for k-tails, since any k-tail has to encompass *all* terms eventually. On the contrary, consider (x_n) with $x_n = n^2$ if n is odd and $x_n = 0$ if n is even. If $n_i = 2i$, then $x_{n_i} = x_{2i} \equiv 0$ is obviously bounded, but (x_n) is unbounded.

Proposition 3.15. Suppose $(x_n)_{i=1}^{\infty}$ converges to x. Then, any subsequence $(x_{n_i})_{i=1}^{\infty}$ also converges to x.

Proof. Suppose $(x_n)_{n=1}^{\infty}$ converges to x. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N, |x_n - x| < \epsilon$. By induction, we have $n_i \geq i$, so $\forall i > N, n_i > i$ and thus $\left|x_{n_i} - x\right| < \epsilon$. The proof is completed. □

This seems pretty straightforward and not all that interesting. But if we think about the contrapositive, we get quite a couple useful corollaries.

Corollary 3.16. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence. If any subsequence of (x_n) diverges, then (x_n) diverges. If two convergent subsequence of (x_n) converge to different values, then (x_n) diverges.

This is a really quick test for which sequences *cannot* converge. For example, to show that $x_n = (-1)^n$ diverges, we can simply note that $x_{2n} = 1$ converges to 1 but $x_{2n-1} = -1$ converges to -1. So by Corollary 3.16, the sequence (x_n) diverges.

Before the end of this section, we will prove the famous Bolzano–Weierstrass Theorem.

Theorem 3.17 (Bolzano–Weierstrass)**.** *Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence. Then, there exists a monotone subsequence $(x_{n_i})_{i=1}^{\infty}$ of (x_n) that is convergent.*

Proof. We say that $n \in \mathbb{N}$ is a *peak* iff $\forall m > n, x_m \le x_n$.

Suppose that (x_n) has infinitely many peaks, $n_1, n_2, \cdots$. Then, $(x_{n_i})_{i=1}^{\infty}$ is a monotone decreasing sequence. Then, by Theorem 3.9, (x_{n_i}) converges.

Suppose that (x_n) has finitely many, but non-zero, peaks, $n_1, \cdots, n_N$. Let $r_1 = n_N + 1$. Since n_N is the final peak, $r_1 > n_N$ is not a peak. Thus, $\exists m > r_1, x_m > x_{r_1}$. Let $r_2 = m > r_1$. We may now repeat the process recursively for $i \in \mathbb{N}$: since n_N is the final peak, $r_i > \cdots > r_1 > n_N$ is not a peak. Thus, $\exists m > r_i, x_m > x_{r_i}$. Let $r_{i+1} = m > r_i$. Thus, $(x_{r_i})_{i=1}^{\infty}$ is a monotone increasing subsequence. We similarly conclude that (x_{r_i}) converges.

Suppose now that (x_n) has no peaks. Then, $\forall n \in \mathbb{N}, \exists m > n, x_m > x_n$. Let $r_1 = 1$. Since $r_1 \in \mathbb{N}$, there exists some $m > r_1$ such that $x_m > x_{r_1}$. Let $r_2 = m \in \mathbb{N}$. We may now repeat the process recursively for $i \in \mathbb{N}$: since $r_i \in \mathbb{N}$, there exists $m > r_i$ such that $x_m > x_{r_i}$. Let $r_{i+1} = m \in \mathbb{N}$. Thus, $(x_{r_i})_{i=1}^{\infty}$ is a monotone increasing subsequence. We similarly conclude that (x_{r_i}) converges. □

To finish this section, let's prove that a sequence is convergent if it is a Cauchy sequence.

Theorem 3.18. *Let $(x_n)_{n=1}^{\infty}$ be a sequence. (x_n) is convergent if and only it is a Cauchy sequence.*

Proof. From Proposition 3.5, we already have the "only if" direction. It suffices to now show the "if" direction holds.

Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence. We first show that (x_n) must be bounded. Fix $\epsilon = 1$. Then, there exists $N \in \mathbb{N}$ such that $\forall m, n > N$, $|x_m - x_n| < 1$. Let $m = N + 1$. Then, $\forall n > m, |x_m - x_n| < 1$, so $x_m - 1 < x_n < x_m + 1$. So the N-tail of the sequence, $(x_n)_{n=N+1}^{\infty}$ is bounded, and thus (x_n) is bounded by Proposition 3.11.

Therefore, by Theorem 3.17, there exists a monotone subsequence $(x_{n_i})_{i=1}^{\infty}$ $(1 \le n_1 < n_2 < \cdots)$. Assume without loss of generality that the subsequence is monotone increasing. The limit $x \in \mathbb{R}$ is further equal to $\sup S$, by Theorem 3.9, where $S := \{x_{n_i} \mid i \in \mathbb{N}\}$. Choose an arbitrary $\epsilon > 0$. Since (x_n) is Cauchy, there exists some $N \in \mathbb{N}$ such that for any $m, n > N$, $|x_m - x_n| < \epsilon/2$. Fix n. Then, for any $m > n > N$, $x_n - \epsilon/2 < x_m < x_n + \epsilon/2$, so the n-tail of (x_m) is bounded within $(x_n - \epsilon/2, x_n + \epsilon/2)$. Note that the limit $x \in (x_n - \epsilon/2, x_n + \epsilon/2)$ also, since some subsequence of (x_{n_i}) (discarding terms before $n + 1$) must be a subsequence of $(x_m)_{m=n+1}^{\infty}$. Therefore, for any $m > n$, we have

$$|x_m - x| = |x_m - x_n + x_n - x| \le |x_m - x_n| + |x_n - x| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which completed the proof. □

3.2 More on Sequence Limits

Limits are exhausting but they're a foundational piece of real analysis. Anyways, let's start with the Squeeze Theorem.

Theorem 3.19 (Squeeze Theorem)**.** *Suppose (x_n), (a_n), and (b_n) are sequences with*

$$a_n \le x_n \le b_n$$

for any $n \in \mathbb{N}$. If (a_n) and (b_n) converge to the same value, then (x_n) too converges to that value.

Proof. Suppose (a_n) and (b_n) both converge to x and $a_n \le x_n \le b_n$ for any $n \in \mathbb{N}$. Suppose $\epsilon > 0$. Fix $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{cases} \forall n > N_1, |a_n - x| < \epsilon, \\ \forall n > N_2, |b_n - x| < \epsilon. \end{cases}$$

Let $N = \max\{N_1, N_2\}$. Then, for any $n > N$, we have $|a_n - x| < \epsilon$, or $x - \epsilon < a_n$, and $|b_n - x| < \epsilon$, or $b_n < x + \epsilon$. Since

$$x - \epsilon < a_n \le x_n \le b_n < x - \epsilon,$$

we conclude that $|x_n - x| < \epsilon$, which completes the proof. □

Similar but equally powerful is the fact that limits, if they exists, preserve non-strict inequalities.

Proposition 3.20. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be convergent sequences with $x_n \le y_n$ for any $n \in \mathbb{N}$. Then, $\lim x_n \le \lim y_n$.

Proof. Let $x = \lim x_n$ and $y = \lim y_n$. Suppose $\epsilon > 0$. Fix $N_1, N_2 \in \mathbb{N}$ such that $\forall n > N_1, |x_n - x| < \epsilon/2$ and $\forall n > N_2, |y_n - y| < \epsilon/2$. With $N := \max\{N_1, N_2\}$, we have $x_n - x < \epsilon/2$ and $y - y_n < \epsilon/2$. Then, adding the two gives $x_n - y_n - x + y < \epsilon$, or $x_n - y_n < x - y + \epsilon$, equivalently. Since $x_n \le y_n$ for all $n \in \mathbb{N}$, we have $0 \le y_x - x_n$ and hence $0 < y - x + \epsilon$ by transitivity. Therefore, $x - y < \epsilon$ for any $\epsilon > 0$, so $x - y \le 0$. Thus, $x \le y$. □

An important corollary immediately follows.

Corollary 3.21. Suppose $(x_n)_{n=1}^{\infty}$ is a convergent sequence. If all its terms are non-negative, then the limit is also non-negative. Further, if the limit is positive, then the k-tail of (x_n) is always positive for some $k \in \mathbb{N}$.

Proof. Suppose (x_n) is a convergent sequence with all terms non-negative. Then, $x_n \ge 0$ for all $n \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} x_n \ge \lim_{n\to\infty} 0 = 0.$$

Suppose (x_n) is a convergent sequence with a positive limit $x := \lim x_n > 0$. Suppose for the sake of contradiction that any k-tail of (x_n) contains some non-positive term $x_{n_k} \le 0$, where $n_k \ge k$. Then, there exists some subsequence $(x_{n_i})_{i=1}^{\infty}$ of (x_n) with all non-positive terms, which has a non-positive limit. However, by Proposition 3.15, $x_{n_i} \to x > 0$, which is a contradiction. The proof is finished. □

We now state the continuity of algebraic operations:

Proposition 3.22. Suppose (x_n) and (y_n) are convergent sequences with limits x and y respectively. Then,

- $(x_n + y_n)$ converges to $x + y$;
- $(x_n - y_n)$ converges to $x - y$;
- $(x_n \cdot y_n)$ converges to $x \cdot y$;
- (x_n/y_n) converges to x/y, provided that (i) $y \neq 0$ and (ii) y_n is never zero for any $n \in \mathbb{N}$.

Let's first prove that the limit of the sum/different of two sequences equals the sum/difference of the respective limits, provided they exist.

Proof. Suppose (x_n) and (y_n) are convergent sequences with limits x and y respectively. Let $\epsilon > 0$. Fix $N_1, N_2 \in \mathbb{N}$ such that $\forall n > N_1, |x_n - x| < \epsilon/2$ and $\forall n > N_2, |y_n - y| < \epsilon/2$. Let $N := \max\{N_1, N_2\}$. Then,

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$$

$$
\begin{aligned}
&\leq |x_n - x| + |y_n - y| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
|(x_n - y_n) - (x - y)| &= |(x_n - x) + (y - y_n)| \\
&\leq |x_n - x| + |y_n - y| \\
&= \epsilon.
\end{aligned}
$$

The proof is finished. □

We now show that the limit of the product of two sequences equals the product of the respective limits, provided they exist.

Proof. Suppose (x_n) and (y_n) are convergent sequences with limits x and y respectively. Suppose $\epsilon > 0$ is given. Let $K := \max\{|x|, |y|, \epsilon/3, 1\}$. Choose $N_1, N_2 \in \mathbb{N}$ such that $\forall n > N_1, |x_n - x| < \epsilon/3K$ and $\forall n > N_2, |y_n - y| < \epsilon/3K$. Then,

$$
\begin{aligned}
|x_n \cdot y_n - x \cdot y| &= |(x_n - x + x)(y_n - y + y) - xy| \\
&= |x_n - x| \cdot |y_n - y| + |y| \cdot |x_n - x| + |x| \cdot |y_n - y| \\
&< \epsilon/3K \cdot \epsilon/3K + K \cdot \epsilon/3K + K \cdot \epsilon/3K \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}
$$

□

Before we prove division, we give the following proposition.

Proposition 3.23. Suppose no term of $(y_n)_{n=1}^{\infty}$ is zero and y_n converges to $y \in \mathbb{R}$. Then, $(1/y_n)$ converges to $1/y$.

Proof. Suppose no term of $(y_n)_{n=1}^{\infty}$ is zero and y_n converges to $y \in \mathbb{R}$. Let $b = \min\{|y|^2 \epsilon/2, |y|/2\} > 0$. Let $\epsilon > 0$. Then, there exists some $N \in \mathbb{N}$ such that $\forall n > N, |y_n - y| < b$. Then, for any $n > N$, $|y_n - y| < b \leq |y|/2$, so

$$
|y| = |y - y_n + y_n| \leq |y_n - y| + |y_n| < \frac{|y|}{2} + |y_n|.
$$

Thus, $|y|/2 < |y_n|$, or equivalently, $1/|y_n| < 2/|y|$. Therefore,

$$
\begin{aligned}
\left|\frac{1}{y_n} - \frac{1}{y}\right| &= \left|\frac{y_n - y}{y \cdot y_n}\right| \\
&< \frac{|y_n - y|}{|y|} \cdot \frac{2}{|y|} \\
&< \frac{2b}{y^2} \\
&< \epsilon.
\end{aligned}
$$

The proof is finished. □

If we multiply (x_n) and $(1/y_n)$, we obtain the proof for the fourth statement.

Now we can establish most convergence tests for sequences.

Proposition 3.24. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence. Suppose $(a_n)_{n=1}^{\infty}$ converges to 0 and further, for some $x \in \mathbb{R}$,

$$|x_n - x| \leq a_n.$$

Then, (x_n) converges to x.

Proof. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence. Suppose $(a_n)_{n=1}^{\infty}$ converges to 0 and further, for some $x \in \mathbb{R}$,

$$|x_n - x| \leq a_n.$$

Suppose $\epsilon > 0$. Fix some $N \in \mathbb{N}$ such that $\forall n > N, |a_n| < \epsilon$. Then, for any $n > N$, we have

$$|x_n - x| \leq a_n < \epsilon.$$

The proof is finished. □

Lemma 3.25 (Ratio Test for Sequences)**.** Let $(x_n)_{n=1}^{\infty}$ be a sequence with no zero terms. Suppose

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists. Then, (x_n) converges to 0 if $L < 1$ and diverges if $L > 1$. If $L = 1$, the ratio test is inconclusive.

Proof. Suppose (x_n) is a sequence with no zero terms and $L = \lim_{n\to\infty} |x_{n+1}| / |x_n|$ exists.

Suppose $L < 1$. Since $|x_{n+1}/x_n| \geq 0$, we conclude from Proposition 3.20 that $L \geq 0$. Choose some arbitrary r such that $0 \leq L < r < 1$. We then compare (x_n) to (r^n). Since $r - L > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N, ||x_{n+1}/x_n| - L| < r - L$. Thus, for $n > N$, $|x_{n+1}/x_n| < r$. Therefore,

$$|x_n| = |x_M| \cdot \frac{x_{M+1}}{x_M} \cdot \frac{x_{M+2}}{x_{M+1}} \cdots \frac{x_n}{x_{n-1}} < |x_M| \cdot \overbrace{r \cdot r \cdots r}^{(n-M)\text{ factors}} = |x_M|\, r^{-M} \cdot r^n.$$

Note that $|x_M|\, r^{-M} r^n$ is a positive sequence tending to 0, so by Proposition 3.24, we conclude that $\lim x_n = 0$.

Suppose $L > 1$. Choose r such that $1 < r < L$. Since $L - r > 0$, there exists some $N \in \mathbb{N}$ such that $\forall n > N, ||x_{n+1}/x_n| - L| < L - r$, so $|x_{n+1}/x_n| > r$. Similarly, for $n > N$, we have

$$|x_n| = |x_M|\, r^{-M} \cdot r^n.$$

Since the sequence $(|x_M|\, r^{-M} \cdot r^n)_{n=1}^{\infty}$ is unbounded, (x_n) must also be unbounded, which therefore diverges according to Proposition 3.7. □

3.3 Limit Superior and Limit Inferior

Let's now talk about the limit superior and the limit inferior.

Definition 3.26. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence. Define the sequences

$$a_n := \sup\{x_k \mid k \geq n\} \quad \text{and} \quad b_n := \inf\{x_k \mid k \geq n\}.$$

Then, when the following limits exist, we define the limit superior of (x_n) as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n$$

and the limit inferior of (x_n) as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} b_n.$$

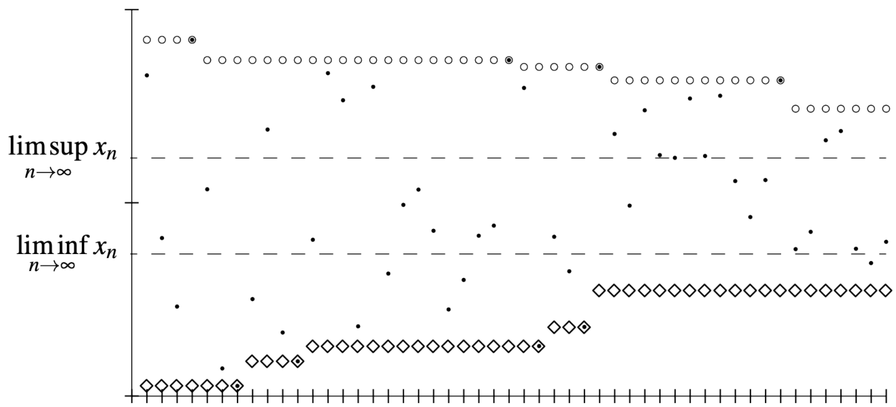

Figure 1: An illustration of the limit superior and the limit inferior. (x_n) is in dots; (a_n) is in circles; (b_n) is in diamonds. This is Fig. 2.5 from [1].

The limit superior and the limit inferior are like weaker ways to describe this "limiting" behavior of a sequence (x_n) (see Fig. 1): it's the "upper bound" and the "lower bound" of the sequence as $n \to \infty$, respectively. As we'll see, the limit superior and the limit inferior of a sequence (x_n) are guaranteed to exist if (x_n) is bounded.

Proposition 3.27. Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence. Let (a_n) and (b_n) be defined as in Definition 3.26. Then,

- $a_n \le x_n \le b_n$ for all $n \in \mathbb{N}$;
- (a_n) is monotone decreasing and (b_n) is monotone increasing;
- The limit superior and the limit inferior of (x_n) exist; that is, (a_n) and (b_n) both converge;
- The limit inferior is no greater than the limit superior; that is, $\liminf x_n \le \limsup x_n$.

Proof. Let $(x_n)_{n=1}^{\infty}$ be bounded. Let $a_n := \sup\{x_k \mid k \ge n\}$ and $a_n := \inf\{x_k \mid k \ge n\}$.

Denote with S the set $\{x_n, x_{n+1}, x_{n+2}, \cdots\}$. Then, $a_n = \sup S$ and $b_n = \inf S$. Thus, $b_n \le x_n \le a_n$.

Choose an arbitrary $n \in \mathbb{N}$. Observe that $\{x_k \mid k \ge n+1\} \subseteq \{x_k \mid k \ge n\}$, so the supremum of the former (i.e., a_{n+1}) is no greater than that of the latter (i.e., a_n). Thus, (a_n) is monotone decreasing. Similarly, we conclude that (b_n) is monotone increasing.

Because each term of (a_n) is the supremum of $\{x_n, x_{n+1}, x_{n+2}, \cdots\}$, we have $x_n \le a_n$. Suppose (x_n) is bounded from below by M; that is, $\forall n \in \mathbb{N}, x_n \ge M$. Then, $a_n \ge M$. Thus, a_n is bounded from below. Since (a_n) is decreasing, it must be bounded from above by the first term a_1. Therefore, (a_n) is bounded. Since (a_n) is monotone decreasing, we conclude that (a_n) converges. Similarly, we deduce that (b_n) converges.

Since $\forall n \in \mathbb{N}, b_n \le a_n$, by Proposition 3.20, we have $\lim b_n \le \lim a_n$, namely, $\liminf x_n \le \limsup a_n$. The proof is completed. □

Now that I think about it, I should've put Bolzano-Weierstrass after all this talk about limit superiors and limit inferiors... but anyways,

Proposition 3.28. Let (x_n) be a bounded sequence. Then there exists a subsequence $(x_{m_i})_{i=1}^{\infty}$ that converges to $\limsup_{n\to\infty} x_n$; similarly, there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ that converges to $\liminf_{n\to\infty} x_n$.

Proof. We define $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ as usual:

$$a_n := \sup\{x_k \mid k \ge n\}; \quad b_n := \inf\{x_k \mid k \ge n\}.$$

Let $a = \lim a_n$ and $b = \lim b_n$ (convergence guaranteed by Proposition 3.27). We construct the subsequence $(x_{n_i})_{i=1}^{\infty}$ inductively. Let $n_1 = 1$. Suppose n_i has been defined for $i = 1, 2, \cdots, k$ ($k \in \mathbb{N}$). Then, for some $n' > n_k$,

$$a_{n_k+1} - x_{n'} < \frac{1}{k+1},$$

because a_{n_k+1} is the supremum of the set $\{x_{n_k+1}, x_{n_k+2}, \cdots\}$, which allows us to use Proposition 2.6. Let $n_{k+1} = n'$, which completes the definition of $(x_{n_i})_{i=1}^{\infty}$.

Note that $a_{n_i} \le a_{n_{i-1}+1}$ for any integer $i \ge 2$: since $n_i > n_{i-1}$, we have $n_i \ge n_{i-1} + 1$, so $a_{n_i} \le a_{n_{i-1}+1}$.

Observe that for any $i = 2, 3, \cdots$,

$$\begin{aligned} \left|a_{n_i} - x_{n_i}\right| &= a_{n_i} - x_{n_i} \\ &\le a_{n_{i-1}+1} - x_{n_i} \\ &< \frac{1}{i}. \end{aligned}$$

Suppose $\epsilon > 0$. By the Archimedean property of real numbers, there exists some $M \in \mathbb{N}$ such that $M \cdot \epsilon > 2$; that is, $\epsilon > 2/M$. Then, fix $N \in \mathbb{N}$ such that $\forall n > N, |a_n - a| = a_n - a < 1/M$. For any $i > \max\{N, M\}$,

$$\begin{cases} 0 \le a_{n_i} - a \le a_i - a < 1/M, \\ \left|a_{n_i} - x_{n_i}\right| < 1/i < 1/M. \end{cases}$$

Therefore,

$$\left|x_{n_i} - a\right| \le \left|x_{n_i} - a_{n_i}\right| + (a_{n_i} - a) < 2/M < \epsilon.$$

The proof is finished. □

If towards infinity, (a_n) and (b_n) defined as usual for some bounded sequence (x_n) converge to the same number, then the sequence converges. This is a useful result since the existence of lim sup's and lim inf's only requires that the sequence is bounded.

Proposition 3.29. Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers. Then, (x_n) converges if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. Further, if (x_n) converges, it must converge to the limit superior and the limit inferior.

Proof. We define $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ as usual:

$$a_n := \sup\{x_k \mid k \ge n\}; \quad b_n := \inf\{x_k \mid k \ge n\}.$$

"If" Direction. Suppose (x_n) is bounded and $\limsup x_n = \liminf x_n$; that is, $\lim a_n = \lim b_n$. Note that for any $n \in \mathbb{N}$, $a_n \le x_n \le b_n$. Therefore, by the Squeeze Theorem (Theorem 3.19), $\lim x_n = \lim a_n = \lim b_n$.

"Only If" Direction. Suppose (x_n) converges to $x \in \mathbb{R}$. For $\epsilon > 0$, fix $N \in \mathbb{N}$ such that $\forall n > N, |x_n - x| < \epsilon/2$. Thus, $x - \epsilon/2 < x_n < x + \epsilon/2$. Define the sets $S_n := \{x_n, x_{n+1}, \cdots\}$ for $n \in \mathbb{N}$. Then, for any $n > N$, $a_n = \sup S_n \ni x_n$, so $a_n \ge x_n > x - \epsilon/2$. Further, observe that $x + \epsilon/2$ is an upper bound of S_n for any $n > N$. Because a_n is the least upper bound of S_n, $a_n \le x + \epsilon/2$. Therefore, $|a_n - x| \le \epsilon/2 < \epsilon$. The proof is finished. □

I really, really hate that whole $\epsilon/2$ shii... It's grossly inelegant. Everyone knows $|a_n - x| \le \epsilon$ literally has no difference from $|a_n - x| < \epsilon$ in this kind of proofs, but whatever.

3.4 Infinite Series

We'll give the usual infinite series definition for convergence.

Definition 3.30. Let $(x_n)_{n=1}^{\infty}$ be a sequence. The formal object

$$x_1 + x_2 + \cdots \quad \text{or} \quad \sum_{n=1}^{\infty} x_n$$

is said to be an infinite series.

Let $S_n := \sum_{i=1}^{n} x_i$. If S_n converges to $x \in \mathbb{R}$, we say that

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} S_n.$$

Otherwise, the formal object named the series is said to diverge.

We'll first talk about geometric series.

Proposition 3.31 (Geometric Series)**.** Suppose $r \in \mathbb{R}$. Then, the infinite series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$. Further, if $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Proof. Define $S_n = \sum_{i=0}^{n} r^n$. When $r = 1$, (S_n) is unbounded and thus diverges. Otherwise, $S_n = (1 - r^n)/(1 - r)$.

The series then converges iff the following limit exists:

$$\lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{1}{1-r} \cdot \lim_{n\to\infty} r^n.$$

By the ratio test (Lemma 3.25), we immediately conclude that $|r| < 1$ implies convergence and $|r| > 1$ implies divergence. If $r = -1$, then

$$S_n = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Clearly, the subsequence (S_{2n}) converges to 1 and (S_{2n+1}) converges to 0. Therefore, by Proposition 3.15, (S_n) diverges. The proof is completed. □

Now, let's state our typical series convergence tests!

Proposition 3.32. Suppose $\sum_{n=1}^{\infty} x_n$ is a convergent infinite series. Then, $\lim_{n\to\infty} x_n = 0$.

The contrapositive of the above is the commonly known divergence test.

Corollary 3.33 (Divergence Test)**.** Suppose $\sum_{n=1}^{\infty} x_n$ is given. If (x_n) either diverges or converges to some non-zero real number, then $\sum x_n$ diverges.

Proof. Let $S_n = \sum_{i=1}^{n} x_i$. We are given that (S_n) converges. Observe that $x_n = S_n - S_{n-1}$ for any $n > 1$. Thus,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0,$$

since Proposition 3.12 implies that (S_n) and (S_{n-1}) converge to the same values. □

Proposition 3.34 (Linearity of Series)**.** Suppose $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent series and $a, b \in \mathbb{R}$. Then, the series $\sum_{n=1}^{\infty}(a \cdot x_n + b \cdot y_n)$ is convergent and converges to

$$\sum_{n=1}^{\infty}(a \cdot x_n + b \cdot y_n) = a \cdot \sum_{n=1}^{\infty} x_n + b \cdot \sum_{n=1}^{\infty} y_n.$$

This comes immediately from the linearity of sequence limits applied to the partial sums S_n's.

Definition 3.35. Suppose $\sum_{n=1}^{\infty} x_n$ is a series. Then, $\sum x_n$ is said to be absolutely convergent (or to converge absolutely) iff $\sum |x_n|$ converges.

Proposition 3.36 (Absolute Convergence)**.** Suppose $\sum x_n$ is absolutely convergent. Then, $\sum x_n$ is convergent.

Proof. Let $S_n := \sum_{i=1}^{n} x_n$ and $S'_n := \sum_{i=1}^{n} |x_n|$. Then, by definition, (S'_n) converges as $n \to \infty$. Thus, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\left|S'_m - S'_n\right| < \epsilon$. Equivalently,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N, n \leq m \Rightarrow (|x_n| + \cdots + |x_m|) < \epsilon.$$

Choose an arbitrary $\epsilon > 0$ and fix $N \in \mathbb{N}$. Then, for any $m, n > N$ with $n \leq m$, we have

$$|x_n + \cdots + x_m| \leq |x_n| + \cdots + |x_m| < \epsilon.$$

Therefore, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for any $m, n > N$, $|S_m - S_n| < \epsilon$. Thus, $\sum x_n$ converges, and the proof is completed. □

3.4.1 Convergence Tests for Series

Proposition 3.37 (Comparison Test)**.** Suppose $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. Then,

- If $\sum y_n$ converges, then $\sum x_n$ converges;
- If $\sum x_n$ is unbounded, then $\sum y_n$ is unbounded.

Proof. Let $S_n := x_1 + \cdots + x_n$ and $S'_n := y_1 + \cdots + y_n$. Then, for any $n \in \mathbb{N}$, we have $0 \leq S_n \leq S'_n$. Further, observe that no terms of either series are negative, so both (S_n) and (S'_n) are monotone increasing.

Suppose $\sum y_n$ converges, or equivalently, (S'_n) converges. Then, by Proposition 3.9, (S'_n) is bounded. Therefore, (S_n) is bounded, and thus $\sum x_n$ converges.

Now suppose instead that $\sum x_n$ is unbounded. Since $S'_n \geq S_n$, (S'_n) is unbounded. Since it is monotone, we similarly conclude that $\sum y_n$ diverges. □

Proposition 3.38 (Limit Comparison Test)**.** Suppose (a_n) and (b_n) are positive-valued sequence. If the limit

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$$

exists and is positive, then $\sum x_n$ and $\sum y_n$either both converge or both diverge.

"Proof left as exercise" lol.

Proposition 3.39 (p-Series)**.** The p-series $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $p > 1$.

Proof. Define $S_n := 1 + 2^{-p} + \cdots + n^{-p}$ for $n \in \mathbb{N}$, which is monotone increasing.

Suppose $p > 1$. We first look at the subsequence (S_{2n+1}). Observe that

$$\begin{aligned} S_{2n+1} &= 1 + \sum_{i=1}^{n} \left(\frac{1}{(2i)^p} + \frac{1}{(2i+1)^p} \right) \\ &< 1 + \sum_{i=1}^{n} \frac{2}{(2i)^p} \\ &= 1 + 2^{1-p} \cdot \sum_{i=1}^{n} i^{-p} \end{aligned}$$

$$
\begin{aligned}
&= 1 + 2^{1-p} \cdot S_n \\
&< 1 + S_n.
\end{aligned}
$$

On the other hand, $S_n < S_{2n+1}$ strictly. So, $S_{2n+1} < 1 + 2^{1-p} \cdot S_n < 1 + 2^{1-p} \cdot S_{2n+1}$. So, $S_n < S_{2n+1} < \dfrac{1}{1 - 2^{1-p}}$. Thus, (S_n) is bounded from above. Since it is also increasing, by Theorem 3.9, (S_n) converges. Thus, $\sum n^{-p}$ converges if $p > 1$.

Divergence for $p \leq 1$ is "left as exercise." ◻

Proposition 3.40 (Root Test for Series). Suppose $\sum_{n=1}^{\infty} x_n$ is given and $L := \limsup_{n\to\infty} |x_n|^{1/n}$ exists. Then, the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Proof. Suppose $\sum x_n$ is given and $L := \limsup |x_n|^{1/n}$ exists.

Suppose $L < 1$. Choose any $r \in (L, 1)$ and fix $N \in \mathbb{N}$ such that $\forall n \geq N, |x_n|^{1/n} < r$, so $|x_n| < r^n$. Thus,

$$
\begin{aligned}
\sum_{n=1}^{\infty} |x_n| &= \sum_{n=1}^{N-1} |x_n| + \sum_{n=N}^{\infty} |x_n| \\
&< \sum_{n=1}^{N-1} |x_n| + \sum_{n=N}^{\infty} r^n \\
&= \sum_{n=1}^{N-1} |x_n| + \frac{r^N}{1-r} \in \mathbb{R}.
\end{aligned}
$$

Thus, $\sum x_n$ converges absolutely.

Now suppose that $L > 1$. We similarly choose $r \in (L, 1)$ and fix $N \in \mathbb{N}$ such that $\forall n \geq N, |x_n| > r^n$. Since $r > 1$, by Lemma 3.25, we conclude that $|x_n|$ is diverges. Thus, it is impossible for (x_n) to converge to 0,[4] and by Proposition 3.33, $\sum x_n$ diverges. The proof is finished. ◻

Proposition 3.41 (Alternating Series Test). Suppose $(x_n)_{n=1}^{\infty}$ is a positive and decreasing sequence that tends to 0. Then, $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ converges.

Proof. Suppose (x_n) is positive and decreasing and tends to 0. Let $S_n := \sum_{k=1}^{n} (-1)^{k+1} x_k$. Clearly, (S_{2n-1}) is a decreasing subsequence bounded from below by S_2 and (S_{2n}) is an increasing sequence bounded from above by S_1. Thus, both subsequences converge. Let $a := \lim S_{2n}$.

We now show that $\lim S_n = a$. Fix $N_1 \in \mathbb{N}$ such that $|S_{2n} - a| < \epsilon/2$ for any $n \geq N_1/2$. Further, fix $N_2 \in \mathbb{N}$ such that $x_{2n+1} < \epsilon/2$ for any $n \geq N_2/2$.

Let $N := \max\{N_1, N_2\}$. Suppose $2n \geq N$. Then, for even terms, $|S_{2n} - a| < \epsilon/2 < \epsilon$. For odd terms,

$$
|S_{2n+1} - a| = |S_{2n} + x_{2n+1} - a| \leq |S_{2n} - a| + |x_{2n+1}| < \epsilon/2 + \epsilon/2 = \epsilon.
$$

Therefore, $\lim S_n = a$, and thus $\sum (-1)^{n+1} x_n$ converges. The proof is complete. ◻

TODO: Series rearrangements, series multiplication (Mertens' theorem), and power series.

4 Functions, Limits, and Continuity

In this section, we'll finally go in to more general functions $f : S \to \mathbb{R}$ ($S \subset \mathbb{R}$).

[4] This requires us to prove that $\lim |x_n| = \lim x_n$, which we omit for brevity.

4.1 Limits of Functions

To talk about limits of functions, it's helpful to first introduce the concept of cluster points. I guess in this way, we can say things like

$$\lim_{x\to 0} f(x) = 1, \quad \text{where } f\colon (0,+\infty) \to \mathbb{R}, x \mapsto x^x.$$

Definition 4.1. Suppose $S \subset \mathbb{R}$ and $c \in \mathbb{R}$ are given. Then, c is said to be a cluster point of S iff any *deleted neighborhood* of c contains at least one point in S; that is, $\forall \epsilon > 0, \exists s \in S\backslash\{c\}, s \in (c-\epsilon, c+\epsilon)$. If any $x \in \mathbb{R}$ is a cluster point of S, then we say that S is dense in $\mathbb{R}$.

For example, for $a < b$, the cluster points of $(a, b]$ are $[a, b]$. The cluster points of $\mathbb{Q}$ are $\mathbb{R}$ (so $\mathbb{Q}$ is dense in $\mathbb{R}$).

We'll try and tie this stuff back to sequences too:

Proposition 4.2. Suppose $S \subset \mathbb{R}$ and $c \in \mathbb{R}$ are given. Then, c is a cluster point of S if and only if there exists a sequence $(x_n)_{n=1}^\infty$ in $S\backslash\{c\}$ that converges to c.

Proof. Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$ be given.

"If" Direction. Suppose $(x_n)_{n=1}^\infty$ is a sequence in $S\backslash\{c\}$ that converges to c. Let $\epsilon > 0$ be arbitrary. Then, fix $N \in \mathbb{N}$ such that $c - \epsilon < x_n < c + \epsilon$ for any $n \geq N$. Let $s := x_N \in S\backslash\{c\}$, which satisfies the required condition.

"Only If" Direction. Suppose c is a cluster point of S. For any $n \in \mathbb{N}$, let $\epsilon := 1/n$ and fix some $s \in S\backslash\{c\}$ such that $s \in (c-\epsilon, c+\epsilon)$. Set $x_n := s$. The definition for (x_n) is complete.

Now suppose $\epsilon > 0$ is arbitrary. By the Archimedean property of real numbers, choose some $N \in \mathbb{N}$ such that $N \cdot \epsilon > 1$. By construction, $|x_n - c| < 1/n$. So, for any $n > N$, $|x_n - c| < 1/n < 1/N < \epsilon$. The proof is finished. □

Definition 4.3. Suppose $S \subset \mathbb{R}$ is given and $c \in \mathbb{R}$ is a cluster point of S. Let $f\colon S \to \mathbb{R}$ be a function. We say that $f(x)$ converges to L as x tends to c, denoted as $\lim_{x\to c} f(x) = L$, iff

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in S\backslash\{c\}, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

We'll omit the proof that the limit is unique, which is easy to show. This justifies our use of the limit *as a number*, when it exists.

Again, we'll tie back the function limit to sequence limits! Note that how if c is a cluster point of S, then there's at least one sequence (x_n) in $S\backslash\{c\}$ that converges to c, from the Proposition above.

Lemma 4.4 (Sequential Limits of Functions)**.** Suppose $S \subset \mathbb{R}$ is given and $c \in \mathbb{R}$ is a cluster point of S. Suppose further that $f\colon S \to \mathbb{R}$ and $L \in \mathbb{R}$ are given. Then, $\lim_{x\to c} f(x) = L$ if and only if every sequence $(x_n)_{n=1}^\infty$ in $S\backslash\{c\}$ that converges to c has $f(x_n) \to L$.

Proof. Let $S \subset \mathbb{R}$ be given and $c \in \mathbb{R}$ a cluster point of S. Suppose $f\colon S \to \mathbb{R}$ and $L \in \mathbb{R}$ are given.

"If" Direction. We use proof by contrapositive. Suppose on the contrary that $f(x)$ does not converges to L as $x \to c$. Fix $\epsilon > 0$ such that for any $\delta > 0$, there exists some $x \in (S\backslash\{c\}) \cap (c-\delta, c+\delta)$ with $|f(x) - L| \geq \epsilon$. We construct $(x_n)_{n=1}^\infty$ as follows. Let $\delta = 1/n > 0$ and fix x as above. Let $x_n := x$, which is bounded in $(c-\delta, c+\delta)$. Thus, $|x_n - c| < \delta = 1/n$. Since $1/n \to 0$, we conclude that $x_n \to c$. However, by construction, $f(x_n) \not\to L$.

"Only If" Direction. Suppose $f(x) \to L$ as $x \to c$. Let $(x_n)_{n=1}^\infty$ be a sequence in $S\backslash\{c\}$ that converges to c. Let $\epsilon > 0$ be given and fix $\delta > 0$ such that for any $x \in S\backslash\{c\}$, $|x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$. Since (x_n) converges to c, fix $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ for any $n > N$. Thus, $\forall n > N, |f(x_n) - L| < \epsilon$. The proof is finished. □

This is actually huge, since basically any property of the limit of functions can be derived with this!

Proposition 4.5 (Preservation of Non-Strict Inequality)**.** Suppose $S \subset \mathbb{R}$ is given and $c \in \mathbb{R}$ is a cluster point of S. Let $f, g\colon S \to \mathbb{R}$ be given. If the limits $L_1 := \lim_{x\to c} f(x)$ and $L_2 := \lim_{x\to c} g(x)$ exist and $f(x) \le g(x)$ for any $x \in S$, then $L_1 \le L_2$.

Proof. Let $(x_n)_{n=1}^{\infty}$ be an arbitrary sequence in $S\backslash\{c\}$ that converges to c. Then, $f(x_n) \to L_1$, $g(x_n) \to L_2$, and $h(x_n) \to L_3$ from Lemma 4.4. On the other hand, we have $f(x_n) \le g(x_n) \le h(x_n)$ for any $n \in \mathbb{N}$. By Proposition 3.20, we have $L_1 \le L_2 \le L_3$. The proof is finished. □

Theorem 4.6 (Squeeze Theorem for Functions)**.** *Suppose $S \subset \mathbb{R}$ is given and c is a cluster point of S. Let $f, g, h\colon S \mapsto \mathbb{R}$ be given such that*

- *$f(x) \le g(x) \le h(x)$ for any $x \in S$;*
- *The limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} h(x)$ exist and are equal.*

Then, $\lim_{x\to c} g(x)$ exists and converges to the same value as $\lim_{x\to c} f(x)$ and $\lim_{x\to c} h(x)$.

Proof. Let $(x_n)_{n=1}^{\infty}$ be an arbitrary sequence in $S\backslash\{c\}$ that converges to c. Note that $f(x_n) \to L := \lim_{x\to c} f(x)$ and $h(x_n) \to L = \lim_{x\to c} h(x)$. The conditions for the squeeze lemma for sequences (Lemma 3.19) are satisfied; thus, $\lim_{n\to\infty} g(x_n) = L$, and by the arbitrary choice of (x_n) we conclude that $\lim_{x\to c} g(x) = L$. The proof is complete. □

Proposition 4.7. Suppose $S \subset \mathbb{R}$ is given and $c \in \mathbb{R}$ is a cluster point of $\mathbb{R}$. If $f, g\colon S \to \mathbb{R}$ are given such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exists, then

- $\lim_{x\to c}[f(x) + g(x)] = [\lim_{x\to c} f(x)] + [\lim_{x\to c} g(x)]$;
- $\lim_{x\to c}[f(x) - g(x)] = [\lim_{x\to c} f(x)] - [\lim_{x\to c} g(x)]$;
- $\lim_{x\to c}[f(x) \cdot g(x)] = [\lim_{x\to c} f(x)] \cdot [\lim_{x\to c} g(x)]$;
- $\lim_{x\to c}[f(x)/g(x)] = [\lim_{x\to c} f(x)]/[\lim_{x\to c} g(x)]$, provided that $\lim_{x\to c} g(x) \ne 0$ and $g(x) \ne 0$ for any $x \in S\backslash\{c\}$.

Proof omitted cuz it follows immediately from the same conclusion for sequences.

4.2 Continuous Functions

Now we have some machinery to talk about continuous functions. While we all know this concept that basically f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$, it's actually a little more intricate than that. From before, c doesn't have to be in the domain, where the concept of continuity of a function (as a predicate) is well-defined over its domain. But besides that, not really that much.

Definition 4.8. Suppose $S \subset \mathbb{R}$, $c \in S$, and $f\colon S \to \mathbb{R}$ are given. Then, f is said to be continuous at c if for any $\epsilon > 0$, there exists some $\delta > 0$ such that $\forall x \in S, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

Well, let's state that relationship between limits and continuity precisely now.

Proposition 4.9. Suppose $S \subset \mathbb{R}$, $c \in S$, and $f\colon S \to \mathbb{R}$ are given. Then,

- f is continuous at c, provided that c is an isolated point of S;
- f is continuous at c if and only if $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = f(c)$, provided that c is a cluster point of S;
- f is continuous at c if and only if $\lim_{n\to\infty} f(x_n) = f(c)$ for any sequence $(x_n)_{n=1}^{\infty}$ in S that converges to c.

None of this is particularly interesting or challenging.

Proof. Suppose $c \in \mathbb{R}$ is an isolated point of $S \subset \mathbb{R}$. Then, fix some $\epsilon_0 \in \mathbb{R}$ such $\forall x \in S \backslash \{c\}, |x - c| \geq \epsilon_0$. Now suppose $\epsilon > 0$ is arbitrary. Let $\delta := \epsilon_0$, so the only possibility that $|x - c| < \delta$ is when $x = c$. Thus, $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$ is true regardless of any further restrictions on f.

Suppose now that $c \in \mathbb{R}$ is a cluster point of S. There are two directions.
"If" Direction. Suppose $\lim_{x \to c} f(x) = f(c)$. For any $\epsilon > 0$, fix $\delta > 0$ such that $\forall x \in S \backslash \{c\}, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. If $x = c$, then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$, and we are done.
"Only If" Direction. Suppose now that f is continuous at c. For any $\epsilon > 0$, fix $\delta > 0$ such that $\forall x \in S, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. We can clearly restrict the possible values of x from S to $S \backslash \{c\}$.

Now suppose f is continuous at c. Let (x_n) be an arbitrary sequence in S that converges to c. Suppose $\epsilon > 0$ is given, and fix $\delta > 0$ such that for any $x \in S$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. Now by virtue that $x_n \to c$, fix $N \in \mathbb{N}$ such that $\forall n > N, |x_n - c| < \delta$. Then, $|f(x_n) - f(c)| < \epsilon$. Therefore, $f(x_n) \to f(c)$.

We will prove the last "only if" statement by contrapositive. Suppose f is not continuous at c. Fix $\epsilon > 0$ such that for any $\delta > 0$, there exists some $x_\delta \in S$ with $|x_\delta - c| < \delta$ but $|f(x_\delta) - f(c)| \geq \epsilon$. We construct the sequence (x_n) as follows: $x_n := x_\delta$ with $\delta = 1/n$ for $n \in \mathbb{N}$. Since $1/n \to 0$, we have $x_n \to c$ by Proposition 3.24. However, $f(x_n) \not\to f(c)$ since $f(x_n)$ is at least distance ϵ from $f(c)$ always. The proof is now complete. □

Now let us now state the continuity of algebraic operations.

Proposition 4.10. Suppose $S \subset \mathbb{R}$ is given. Let $f, g \colon S \to \mathbb{R}$ be continuous functions. Then,

- $f + g$ is continuous;
- $f - g$ is continuous;
- $f \cdot g$ is continuous;
- f/g is continuous, provided that $\forall x \in S, g(x) \neq 0$.

I'll skip the proof since this follows straight from Proposition 4.7.

Now, the limit of a composition. This is a perfect demonstration of the elegance of the sequential characterization of the limit of a function.

Proposition 4.11. Let $A, B \subset \mathbb{R}$, $f \colon B \to \mathbb{R}$, $g \colon A \to B$, and $c \in A$ be given. If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Proof. Let $(x_n)_{n=1}^{\infty}$ be an arbitrary sequence in A that converges to c. Then, $g(x_n) \to g(c)$, and $f(g(x_n)) \to f(g(c))$. The proof is finished. □

Niceeeee. Finally,

Proposition 4.12. Every polynomial is continuous over $\mathbb{R}$; that is, $f \colon \mathbb{R} \to \mathbb{R}$ as defined by

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

is continuous, where $a_0, \cdots, a_n$ are constants and $n \in \mathbb{Z}_{\geq 0}$ is given.

Proof. Suppose $c \in S$ is given. If c is a cluster point of S,

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} \left(a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \right) \\ &= a_d \lim_{x \to c} x^d + a_{d-1} \lim_{x \to c} x^{d-1} + \cdots + a_0 \end{aligned}$$

$$= a_d c^d + a_{d-1} c^{d-1} + \cdots + a_0 \qquad \text{(By Continuity of } \lambda x.x \text{ and Prop. 4.10)}$$
$$= f(c).$$

Otherwise, f is trivially continuous at c. □

4.3 Min–Max and The Intermediate Value Theorem

It is only apt that the IVT has its own section! Before we prove that, we need some useful propositions about continuous functions.

Proposition 4.13. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then, f is bounded.

Proof. Suppose on the contrary that f is not bounded. For any $n \in \mathbb{N}$, fix $x_n \in [a, b]$ such that $|f(x_n)| > n$, so $(f(x_n))$ is unbounded and thus divergent by Proposition 3.7. However, (x_n) is bounded, so we may choose a convergent subsequence (x_{n_k}) by Theorem 3.17; on the other hand, $f(x_{n_k})$ diverges as a subsequence, a contradiction to Proposition 4.9. □

We can now talk about the concept of maxima and minima.

Definition 4.14. Let $S \subset \mathbb{R}$ and $x_0 \in S$ be given and suppose $f : S \to \mathbb{R}$ is continuous. f is said to attain an absolute maximum at x_0 iff $\forall x \in S, f(x) \le f(x_0)$. Similarly, f is said to attain an absolute minimum at x_0 iff $\forall x \in S, f(x) \ge f(x_0)$.

Complementarily, f is said to attain a relative maximum (resp. minimum) at x_0 iff f attains an absolute maximum (resp. minimum) at x_0 when its restricted is restricted to some open interval containing x_0. That is, f is said to attain an absolute maximum at x_0 iff $\exists \delta > 0, \forall x \in S \cap (x_0 - \delta, x_0 + \delta), f(x) \le f(x_0)$. Similarly, f is said to attain an absolute minimum at x_0 iff $\exists \delta > 0, \forall x \in S \cap (x_0 - \delta, x_0 + \delta), f(x) \ge f(x_0)$.

By the way, the x_0 isn't always unique. It might not exist too!

Theorem 4.15 (Min-Max)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then, f has both an absolute maximum and an absolute minimum.*

Proof. By Proposition 4.13, $M_\uparrow := \sup f([a, b]) \in \mathbb{R}$. Then, for any $n \in \mathbb{N}$, fix $x_n \in [a, b]$ such that $M_\uparrow - f(x_n) < \epsilon := 1/n$ by Proposition 2.6. We then obtain (x_n) with $f(x_n) \to M_\uparrow$ by definition. Since (x_n) is bounded (in $[a, b]$), we extract a convergent subsequence (x_{n_k}), and $f(x_{n_k})$ still converges to $M_\uparrow$ by Proposition 3.15.

Now observe that $a \le x_{n_k} \le b$, so the limit $x^* := \lim_{k\to\infty} x_{n_k}$ is still confined within the interval: $a \le \lim x_{n_k} \le b$, so $x^* \in [a, b]$. Therefore, by definition, f attains an absolute maximum at x^*. A similar argument ensues for the minimum. □

We can now state Bolzano's intermediate value theorem! Weirdly enough, we'll do this with the bisection method and *show* that it converges. We're manually constructing that zero, basically. I guess whatever works works.

Theorem 4.16 (Bolzano)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) \ne f(b)$. Then, for any y strictly between $f(a)$ and $f(b)$, there exists some $c \in (a, b)$ such that $f(c) = y$.*

Proof. Without loss of generality, suppose $f(a) < 0$ and $f(b) > 0$, and suppose $y = 0$.

We begin by constructing two sequences simultaneously: $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$. Begin with $a_0 = a$ and $b_0 = b$. Then, inductively, suppose a_n and b_n have been defined ($n \in \mathbb{Z}_{\ge 0}$). Then,

- Let $t = (a_n + b_n)/2$;
- If $f(t) \ge 0$, then let $a_{n+1} := a_n$ and $b_{n+1} := t$;
- Otherwise, let $a_{n+1} := t$ and $b_{n+1} := b_n$.

Define $x_n = (a_n + b_n)/2$ for $n \in \mathbb{Z}_{\geq 0}$. Clearly, $a_n < x_n < b_n$ always, and $b_n - a_n = (b-a) \cdot 2^{-n}$. Further, (a_n) is monotone increasing and (b_n) is monotone decreasing by construction.

Note that (a_n) is bounded from above by b and (b_n) is bounded from below by a, so both sequences are guaranteed to converge by the monotone convergence theorem (Theorem 3.9). By the squeeze theorem (Theorem 3.19), then, x_n converges, and by Proposition 3.20 we assert that $x^* := \lim_{n\to\infty} x_n$ is confined in $[a, b]$.

Now, $f(a_n) \leq 0$ and $f(b_n)$ for any $n \in \mathbb{Z}_{\geq 0}$, so $\lim f(a_n) \leq 0 \leq \lim f(b_n)$. But the limits on the two ends must be $f(\lim a_n)$ and $f(\lim b_n)$ by Lemma 4.4, and so $f(x^*) = \lim f(x_n) = \lim f(a_n) = \lim f(b_n) = 0$. So x^* is a zero of f. □

We now give a short introduction to uniform continuity of a function.

Definition 4.17. Suppose $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ are given. f is said to be uniformly continuous iff for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in S$ such that $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

So basically it needs a δ that doesn't depend on the c.

Theorem 4.18 (Heine–Cantor)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is uniformly continuous.*

Proof. We prove this theorem by the contrapositive. Let $f : [a, b] \to \mathbb{R}$ be given and suppose on the contrary that f is not uniformly continuous. Fix $\epsilon > 0$ such that for all $\delta > 0$, there exist $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.[5]

Fix two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ thereby such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$. By the Bolzano-Weierstrass theorem (Proposition 3.17), we may extract a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n). Let $c := \lim x_{n_k} \in [a, b]$ by Proposition 3.20. Then, for an arbitrary $k \in \mathbb{N}$,

$$\left|y_{n_k} - c\right| = \left|y_{n_k} - x_{n_k} + x_{n_k} - c\right| \leq \left|x_{n_k} - y_{n_k}\right| + \left|x_{n_k} - c\right| \leq \frac{1}{n_k} + \left|x_{n_k} - c\right| \to 0.$$

Thus, $\lim y_{n_k} = c$ as well by Proposition 3.24.

However, $\left|f(x_{n_k}) - f(y_{n_k})\right| \geq \epsilon$ by construction and therefore cannot converge to a common value.[6] Thus, by Proposition 4.9, f is not continuous. The proof is complete. □

5 The Derivative

Let's start right off the bat with the definition.

Definition 5.1. Suppose $I \subset \mathbb{R}$ is an interval and let $f : I \to \mathbb{R}$ and $c \in I$ be given. The derivative of f at c, denoted as $f'(c)$, is defined as the following limit.

$$f'(c) := \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

If the limit above exists, then f is said to be differentiable at c.[7] If f is differentiable at all points in I, then f is said to be differentiable. f is said to be differentiable on $S \subset I$ if it is differentiable at all points in S.

The first thing we learn about the derivative is:

Proposition 5.2. Suppose $I \subset \mathbb{R}$ is an interval and let $f : I \to \mathbb{R}$ and $c \in I$ be given. If f is differentiable at c, then f is continuous at c.

Proof. Note that $\lim_{x\to c}(x - c) = 0$ by Proposition 4.12. Therefore, by Proposition 3.22, the following limit exists and is

$$\lim_{x\to c} (f(x) - f(c)) = \left(\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\right) \cdot \left(\lim_{x\to c} x - c\right) = f'(c) \cdot 0 = 0.$$

[5] That is, "$f(x)$ and $f(y)$ can be made sufficient far apart (by at least some fixed $\epsilon > 0$) when the distance between x and y is made arbitrarily small."

[6] To show this, suppose for the sake of contradiction that the common limit exists. Then, $f(x_{n_k}) - f(y_{n_k}) \to 0$ by Proposition 3.22, which contradicts the definition of the limit being 0.

[7] I understand that this differs for a multivariable real-valued function.

Again, since $\lim_{x \to c} f(c) = f(c)$ by Proposition 4.12, the following limit exists and is

$$\lim_{x \to c} f(x) = \left(\lim_{x \to c} f(x) - f(c) \right) + \left(\lim_{x \to c} f(c) \right).$$

We take for granted the fact that $c \in I$ is necessarily a cluster point of I. Therefore, by Proposition 4.9, f is continuous at c. □

Now linearity.

Proposition 5.3. Suppose $I \subset \mathbb{R}$ is an interval and let $f, g \colon I \to \mathbb{R}$ be differentiable at a given point $c \in I$. Then,

- $f + g$ is differentiable at c, and $(f + g)'(c) = f'(c) + g'(c)$;
- For any $\alpha \in \mathbb{R}$, $\alpha \cdot f$ is differentiable at c, and $(\alpha \cdot f)'(c) = \alpha \cdot f'(c)$.

Proof. This is an immediate result from Proposition 4.10. The following limit exists and is

$$(f + g)'(c) = \lim_{x \to c} \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c} = \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) + \left(\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right) = f'(c) + g'(c).$$

Similarly,

$$(\alpha \cdot f)'(c) = \lim_{x \to c} \frac{\alpha \cdot f(x) - \alpha \cdot f(c)}{x - c} = \alpha \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \alpha \cdot f'(c).$$

The proof is finished. □

We now give the product rule, the quotient rule, and the chain rule. We also include in Appendix **??** a rigorous definition of exponential and trigonometric functions with results up to this point.

Proposition 5.4 (Chain Rule)**.** Let $I_1, I_2 \subset \mathbb{R}$ be intervals and suppose $f \colon I_2 \to \mathbb{R}$, $g \colon I_1 \to I_2$, and $c \in I_1$ are given. If g is differentiable at c and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c).$$

Proof. Let $d \coloneqq g(c) \in I_2$ and define $u \colon I_2 \to \mathbb{R}$ and $v \colon I_1 \to I_2$ as

$$u(y) \coloneqq \begin{cases} \dfrac{f(y) - f(d)}{y - d}, & \text{if } y \neq d, \\ f'(d), & \text{otherwise} \end{cases} \quad \text{and} \quad v(x) \coloneqq \begin{cases} \dfrac{g(x) - g(c)}{x - c}, & \text{if } x \neq c, \\ g'(c), & \text{otherwise} \end{cases}$$

We take for granted the fact that c and d are cluster points in I_1 and I_2 respectively. Since f is differentiable at d, we have

$$f'(d) = \lim_{y \to d} \frac{f(y) - f(d)}{y - d},$$

and hence u is continuous at d. By a similar argument, v is continuous at c.

Observe that $f(y) - f(d) = u(y) \cdot (y - d)$ for all $y \in I_2$ and $g(x) - g(c) = v(x) \cdot (x - c)$ for all $x \in I_1$. Thus, for all $x \in I_1 \backslash \{c\}$,

$$f(g(x)) - f(g(c)) = u(g(x)) \cdot (g(x) - g(c)) = u(g(x)) \cdot v(x) \cdot (x - c).$$

Therefore, by continuity and Proposition 4.11, the following limit exists and is

$$(f \circ g)'(c) = \lim_{x \to c} \frac{(f \circ g)(x) - (f \circ g)(c)}{x - c} = \lim_{x \to c} u(g(x)) \cdot v(x) = f'(g(c)) \cdot g'(c).$$

□

Proposition 5.5 (Product Rule)**.** Let $I \subset \mathbb{R}$ be an interval and suppose $f, g\colon I \to \mathbb{R}$ are both differentiable at a given point $c \in I$. Then, $f \cdot g$ is differentiable at c and

$$(f \cdot g)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c).$$

Proof. Observe that

$$\begin{aligned}
(f \cdot g)'(c) &= \lim_{x \to c} \frac{f(x) \cdot g(x) - f(c) \cdot g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x) \cdot g(x) - f(x) \cdot g(c) + f(x) \cdot g(c) - f(c) \cdot g(c)}{x - c} \\
&= \left(\lim_{x \to c} f(x)\right) \cdot \left(\lim_{x \to c} \frac{g(x) - g(c)}{x - c}\right) + \left(\lim_{x \to c} g(c)\right) \cdot \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right) && \text{(Prop. 4.7)} \\
&= f'(c) \cdot g(c) + f(c) \cdot g'(c).
\end{aligned}$$

The proof is finished. □

Corollary 5.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a degree-d polynomial ($d \in \mathbb{Z}_{\geq 0}$); that is,

$$f(x) = a_0 + \sum_{n=1}^{d} a_n x^n$$

for real constants $a_0, a_1, \cdots, a_n \in \mathbb{R}$. Then, f is differentiable and

$$f'(x) = \sum_{n=1}^{d} n a_n x^{n-1}.$$

This corollary is obvious from induction.

Proposition 5.7 (Quotient Rule)**.** Let $I \subset \mathbb{R}$ be an interval and suppose $f, g\colon I \to \mathbb{R}$ are both differentiable at a given point $c \in I$, and $g(c) \neq 0$. Then, f/g is differentiable at c and

$$(f/g)'(c) = \frac{f'(c) \cdot g(c) - f(c) \cdot g'(c)}{g(c)^2}.$$

This proof is "left as exercise."

Let's now talk about two mean-value theorems.

Theorem 5.8 (Rolle's Mean-Value Theorem)**.** *Suppose $f\colon [a, b] \to \mathbb{R}$ is a continuous function, differentiable on (a, b), such that $f(a) = f(b)$. Then, there exists some $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Let $m := \inf f([a, b])$ and $M := \sup f([a, b])$. If $m = M$, then $f([a, b])$ is a singleton, and hence f is a constant function. Let $c = a$, then $f'(c) = \lim(f(x) - f(c))/(x - c) = \lim 0/(x - c) = 0$. We now suppose $m < M$.

Choose $c \in [a, b]$ such that $f(c) = M$ by Theorem 4.15. We first suppose $c \in (a, b)$. Then, for all $x \in [a, c)$, $f(x) - f(c) \leq 0$ and $x - c < 0$, so

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

For all $x \in (c, b]$, $f(x) - f(c) \leq 0$ and $x - c > 0$, so

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

Thus, $f'(c) = 0$.

If instead $c \in \{a, b\}$, then $f(a) = f(b) = f(c)$. Since $m < M$ strictly, an absolute minimum must be achieved at some $c' \in (a, b)$, for which the same argument applies with the maximum replaced by the minimum. □

Theorem 5.9 (Lagrange's Mean-Value Theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function differentiable on (a, b). Then, there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $m := (f(b) - f(a))/(b - a)$ and define $f^* : [a, b] \to \mathbb{R}$ as $f^*(x) := f(x) - m(x - a) - f(a)$. Then, $f^*(a) = f^*(b)$, and thus there exists c by Theorem 5.8 such that $(f^*)'(c) = 0$. Then, by Propositions 5.3 and 5.6,

$$f'(c) = (f^*)'(c) + m = \frac{f(b) - f(a)}{b - a}.$$

The proof is finished. □

Let's now get into one of the most useful aspects of the derivative: its relationship with monotonocity and local extrema.

Proposition 5.10. Suppose $I \subset \mathbb{R}$ is given and $f : I \to \mathbb{R}$ is differentiable. Then, f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

Proof. Recall that f is said to be increasing if $f(x) \leq f(y)$ for all $x, y \in I$ with $x < y$.

"If" Direction. Suppose on the contrary that f is not increasing. Fix $x, y \in I$ such that $x < y$ but $f(x) > f(y)$. Then, applying Lagrange's mean-value theorem (Theorem 5.9) to $f|_{[x,y]}$, we have some $c \in (x, y)$ such that $f'(c) = (f(y) - f(x))/(y - x) < 0$.

"Only If" Direction. Let f be increasing, and suppose $c \in I$ are given. Then, for all $x \in I \backslash \{c\}$,

$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

Thus, by Proposition 3.20,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

The proof is complete. □

Lemma 5.11 (Fermat)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). If f attains a local maximum or a local minimum at $c \in (a, b)$, then $f'(c) = 0$.

Proof. We first consider the case of maximum. Fix $\delta > 0$ such that $f(x) \leq f(c)$ for all $x \in [a, b] \cap (c - \delta, c + \delta) \backslash \{c\}$.

For any $x \in [a, b] \cap (c - \delta, c)$, $f(x) - f(c) < 0$ and $x - c < 0$. Hence,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0.$$

For any $x \in [a, b] \cap (c, c + \delta)$, $f(x) - f(c) < 0$ and $x - c > 0$. Hence,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Thus, $f'(c) = 0$. The case of minimum follows the same logic. The proof is finished. □

The intermediate value property is surprisingly easy to prove.

Proposition 5.12 (Intermediate Value Property for Derivatives)**.** Let $f\colon [a,b] \to \mathbb{R}$ be differentiable.[8] Then, for any $y \in (\min\{f'(a), f'(b)\}, \max\{f'(a), f'(b)\})$, there exists some $x \in (a,b)$ such that $f'(x) = y$.

Proof. Without loss of generality, suppose $f'(a) < f'(b)$. Define $g\colon [a,b] \to \mathbb{R}$ as $g(x) \coloneqq f(x) - yx$. Then, g is differentiable with $g'(x) = f'(x) - y$. In particular, g is continuous, and by Theorem 4.15, g attains minimum value at some $x_0 \in [a,b]$.

We now show that $x_0 \neq a$. Since $g'(a) = f'(a) - y < 0$—that is,

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} < 0,$$

for $\epsilon \coloneqq -g'(a)$, fix $\delta > 0$ such that for any $x \in (a, a+\delta) \subset (a,b]$, $\left|\frac{g(x)-g(a)}{x-a} - g'(a)\right| < \epsilon = -g'(a)$. Hence, $\frac{g(x)-g(a)}{x-a} < g'(a) - g'(a) = 0$. Since $x \in (a,b]$, $x - a > 0$, so $g(x) < g(a)$. Thus, g cannot attain a minimum at a.

Similarly, $x_0 \neq b$: $g'(b) > 0$, so there exists some $\delta > 0$ such that $\frac{g(x)-g(b)}{x-b} > 0$ for all $x \in (b-\delta, b)$. Hence, $g(x) < g(b)$, so g cannot attain a minimum at b.

Thus, $x_0 \in (a,b)$, and Proposition 5.3 justifies $f'(x_0) = y$. The proof is finished. □

6 The Riemann Integral

I knew I would hate integrals back in high school and I dang right do! The details are so horrendous... But, they're graspable.

Definition 6.1. A partition of a closed interval $[a,b]$ is a finite subset of $[a,b]$ that contains $\{a,b\}$. The collection of all partitions of $[a,b]$ is denoted as $\mathcal{P}[a,b]$ (this is my own notation!).

The elements of a partition $P \in \mathcal{P}[a,b]$ can always be sorted uniquely as $a = x_0 < x_1 < \cdots < x_n = b$, so $|P| = n+1$. The plus one is there because we're really talking about the endpoints.

We will frequently use the following symbols that I believe deserve their own definition.

Definition 6.2. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function and $P \in \mathcal{P}[a,b]$. Suppose $P = \{x_0, \cdots, x_n\}$ where $a = x_0 < \cdots < x_n = b$, and let $\Delta x_i \coloneqq x_i - x_{i-1}$ for $i = 1, 2 \cdots, n$. Let $M_i \coloneqq \sup f([x_{i-1}, x_i])$ and $m_i \coloneqq \inf f([x_{i-1}, x_i])$ $(i = 1, 2, \cdots, n)$. We define the upper and lower Darboux sums:

$$\begin{cases} U(P,f) \coloneqq \displaystyle\sum_{i=1}^{n} M_i \cdot \Delta x_i, \\ L(P,f) \coloneqq \displaystyle\sum_{i=1}^{n} m_i \cdot \Delta x_i. \end{cases}$$

We can now finally define the Darboux integral, which turns out to be equivalent to the Riemann integral. I still don't know how to prove that, but it is sufficiently strong. Plus, the sup and the inf are nice.

Definition 6.3. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function. We define the upper and lower Darboux integrals as follows:

$$\begin{cases} \displaystyle\overline{\int_a^b} f \coloneqq \inf_{P \in \mathcal{P}[a,b]} U(P,f), \\ \displaystyle\underline{\int_a^b} f \coloneqq \sup_{P \in \mathcal{P}[a,b]} L(P,f). \end{cases}$$

If the two integrals are equal, then we say that f is Riemann integrable, and the integral is defined as the common value:

$$\int_a^b f \coloneqq \overline{\int_a^b} f = \underline{\int_a^b} f.$$

[8] We require that the derivative exist on the endpoints of the domain.

We denote with $\mathcal{R}[a,b]$ the set of all Riemann integrable function from $[a,b]$ to $\mathbb{R}$.

We define[9] $f \in \mathcal{R}[a,a]$ for any $f\colon S \to \mathbb{R}$ with $a \in S$. If $f \in \mathcal{R}[a,b]$, then we also define

$$\int_b^a f := -\int_a^b f.$$

I'm pretty certain that for any bounded function, the upper and lower Darboux integrals always exist.

Proposition 6.4. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function and $f([a,b]) \subset [m,M]$. Then, for all $P \in \mathcal{P}[a,b]$,

$$m \cdot (b-a) \le L(P,f) \le U(P,f) \le M \cdot (b-a).$$

Proof. Observe that

$$m \cdot (b-a) = \sum_{i=1}^{n} m \cdot \Delta x_i \le \sum_{i=1}^{n} m_i \cdot \Delta x_i = L(P,f).$$

Similarly,

$$L(P,f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i \le \sum_{i=1}^{n} M_i \cdot \Delta x_i = U(P,f),$$

and

$$U(P,f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i \le \sum_{i=1}^{n} M \cdot \Delta x_i = M \cdot (b-a).$$

The proof is complete. □

This proof above uses a couple facts: the infimum is less than or equal to the supremum; the infimum of a subset is at least the infimum of the superset.

We cannot move forward without talking about refinements.

Definition 6.5. Suppose $P, P^* \in \mathcal{P}[a,b]$. We say that P^* is a refinement of P iff $P \subset P^*$.

Proposition 6.6. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function and let $P, P^* \in \mathcal{P}[a,b]$ be given. If P^* is a refinement of P, then

$$L(P,f) \le L(P^*,f) \le U(P^*,f) \le U(P,f).$$

Proof. Let $P = \{x_0, \cdots, x_n\}$ such that $a = x_0 < \cdots < x_n = b$ and let $P^* = \{\tilde{x}_0, \cdots, \tilde{x}_\ell\}$ such that $a = \tilde{x}_0 < \cdots < \tilde{x}_\ell = b$. For any $i \in \{0, \cdots, n\}$ define $n_i \in \{0, \cdots, \ell\}$ such that $x_i = \tilde{x}_{n_i}$. Then,

$$L(P,f) = \sum_{i=1}^{n} \inf f([x_{i-1}, x_i] \cdot \Delta x_i = \sum_{i=1}^{n} \sum_{j=n_{i-1}+1}^{n_i} \inf f([x_{i-1}, x_i]) \cdot \Delta \tilde{x}_j \le \sum_{i=1}^{n} \sum_{j=n_{i-1}+1}^{n_i} \inf f([\tilde{x}_{j-1}, \tilde{x}_j]) \cdot \Delta \tilde{x}_j = L(P^*, f).$$

By similar reasoning, $U(P^*,f) \le U(P,f)$, and the middle inequality follows from Proposition 6.4. □

Also, a coarse bound for the integral is given here.

Proposition 6.7. Suppose $f \in \mathcal{R}[a,b]$ and $f([a,b]) \subset [m,M]$. Then,

$$m \cdot (b-a) \le \int_a^b f \le M \cdot (b-a).$$

[9]This *is* define, not redefine, because $\mathcal{P}[a,a] \triangleq \{\{a\}\}$ means that neither $U(P,f)$ nor $L(P,f)$ is well-defined for any (or rlly jus the unique) $P \in \mathcal{P}[a,b]$: $n = 0$, so the sum from $i = 1$ to n is undefined.

Proof. Let $P \in \mathcal{P}[a,b]$ be an arbitrary partition of $[a,b]$. Then, by Proposition 6.4, we have

$$m \cdot (b-a) \leq L(P,f) \quad \text{and} \quad U(P,f) \leq M \cdot (b-a).$$

Taking the supremum for the inequality on the left and the infimum on the right,

$$m \cdot (b-a) \leq \int_a^b f \leq M \cdot (b-a).$$

The proof is finished. □

I should really state that the integral is bounded by any lower and upper Darboux sum.

Proposition 6.8. Suppose $f \in \mathcal{R}[a,b]$ and $P \in \mathcal{P}[a,b]$. Then, $L(P,f) \leq \int_a^b f \leq U(P,f)$.

Proof. Well, the integral is both the sup and the inf. We're done. □

We also state an important equivalent characterization of Riemann integrability based on ϵ.

Proposition 6.9. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function. Then, $f \in \mathcal{R}[a,b]$ if and only if for all $\epsilon > 0$, there exists some partition $P \in \mathcal{P}[a,b]$ such that

$$U(P,f) - L(P,f) < \epsilon.$$

Proof. **"If" Direction.** Let $\epsilon > 0$ be given. Then, $0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P,f) - L(P,f) < \epsilon$, so

$$\overline{\int_a^b} f = \underline{\int_a^b} f,$$

and hence $f \in \mathcal{R}[a,b]$.

"Only If" Direction. Suppose $f \in \mathcal{R}[a,b]$. Let $\epsilon > 0$ be given, and denote with I the integral $\displaystyle\int_a^b f$. Choose partitions $P, Q \in \mathcal{P}[a,b]$ such that

$$U(P,f) - I < \epsilon/2 \quad \text{and} \quad I - L(P,f) < \epsilon/2.$$

Adding the two then gives $U(P,f) - L(Q,f) < \epsilon$. Now let $P^* := P \cup Q \in \mathcal{P}[a,b]$, which is a refinement of both P and Q. Therefore, by Proposition 6.6,

$$U(P^*,f) - L(P^*,f) \leq U(P,f) - L(Q,f) < \epsilon.$$

The proof is complete. □

This also translates to a sequential characterization. Note that here n denotes the index of the sequence, not the number of sub-intervals corresponding to a partition.

Proposition 6.10. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function. Then, $f \in \mathcal{R}[a,b]$ if and only if there exists a sequence of partition $\{P_n\}_{n=1}^{\infty}$ of $[a,b]$ such that

$$\lim_{n\to\infty} \big(U(P_n,f) - L(P_n,f)\big) = 0.$$

If so, then $\displaystyle\int_a^b = \lim_{n\to\infty} U(P_n,f) = \lim_{n\to\infty} L(P_n,f)$.

Proof. **"If" Direction.** Let $\epsilon > 0$ be given. Fix $N \in \mathbb{N}$ such that for all $n > N$, $U(P_n,f) - L(P_n,f) < \epsilon$. In particular, $U(P_{N+1},f) - L(P_{N+1},f) < \epsilon$, which implies integrability by Proposition 6.9.

"Only If" Direction. For any $n \in \mathbb{N}$, fix $P_n \in \mathcal{P}[a,b]$ such that $U(P_n,f) - L(P_n,f) < 1/n$. Then, by Proposition 3.24, $\lim_{n\to\infty} \big(U(P_n,f) - L(P_n,f)\big) < \epsilon$.

Denote with I the integral $\int_a^b f$, and observe that $L(P_n, f) \leq I \leq U(P_n, f)$ for all $n \in \mathbb{N}$. Therefore, by Proposition 3.20,

$$\lim_{n\to\infty} L(P_n, f) \leq I \leq \lim_{n\to\infty} U(P_n, f).$$

Since both limits tend to the same value, we conclude that

$$I = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f).$$

The proof is complete. ◻

6.1 Additivity, Linearity, and Conditions of Convergence

Here, we use the word "additivity" to talk about how integrals can be broken down to integrals on sub-intervals.

Proposition 6.11. Suppose $f\colon [a, c] \to \mathbb{R}$ is bounded and $b \in (a, c)$. Then,

$$\begin{cases} \underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f, \\ \overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f. \end{cases}$$

Proof. First, suppose $P_1 \in \mathcal{P}[a, b]$ and $P_2 \in \mathcal{P}[b, c]$ are arbitrary. Then, $P := P_1 \cup P_2 \in \mathcal{P}[a, c]$. Therefore,

$$\underline{\int_a^c} f \geq L(P, f) = L(P_1, f) + L(P_2, f).$$

Taking the supremum on the right hand side over P_1 and P_2, we obtain

$$\underline{\int_a^c} f \geq \sup_{P_1} L(P_1, f) + \sup_{P_2} L(P_2, f) = \underline{\int_a^b} f + \underline{\int_b^c} f.$$

Now suppose instead $P \in \mathcal{P}[a, c]$ is arbitrary. Define $P' := P \cup \{b\} \in \mathcal{P}[a, c]$, which is a refinement of P, and let $P_1 := P' \cap [a, b] \in \mathcal{P}[a, b]$ and $P_2 := P' \cap [b, c] \in \mathcal{P}[b, c]$. Therefore,

$$L(P, f) \leq L(P', f) = L(P_1, f) + L(P_2, f) \leq \underline{\int_a^b} f + \underline{\int_b^c} f.$$

Taking the supremum on the left hand side over P, we obtain

$$\underline{\int_a^c} f \leq \underline{\int_a^b} f + \underline{\int_b^c} f.$$

Thus,

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f.$$

The same argument follows for the upper Darboux integrals, from which we conclude

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f.$$

The proof is complete. ◻

So this one is clever since we avoid equalities, but utilize all the inequalities that we can leverage. Also note that the construction of P', by adding b as a partition point (if it's not already in P ig), we can split the sums directly. And the inequalities for refined partitions (Proposition 6.6) let us complete the transitions.

Proposition 6.12. Suppose $f \in \mathcal{R}[a,c]$, and $b \in (a,c)$. Then, $f|_{[a,b]} \in \mathcal{R}[a,b]$ and $f|_{[b,c]} \in \mathcal{R}[b,c]$, and

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof. By Proposition 6.11, we have

$$\int_a^c f = \begin{cases} \underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f, \\ \overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f. \end{cases}$$

Subtracting the RHS of the top equation from the RHS of the bottom equation, we have

$$\left(\overline{\int_a^b} f - \underline{\int_a^b} f\right) + \left(\overline{\int_b^c} f - \underline{\int_b^c} f\right) = 0.$$

Since both parenthesized expressions above are necessarily non-negative, they must both be zero. Therefore,

$$\overline{\int_a^b} f = \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_b^c} f = \underline{\int_b^c} f.$$

Thus, $f|_{[a,b]} \in \mathcal{R}[a,b]$ and $f|_{[b,c]} \in \mathcal{R}[b,c]$. Further,

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

The proof is finished. □

Corollary 6.13. Suppose $f \in \mathcal{R}[a,d]$ and $a \le b \le c \le d$. Then, $f|_{[b,c]} \in \mathcal{R}[b,c]$.

Proof. First, we have $f|_{[a,b]} \in \mathcal{R}[a,b]$ and $f|_{[b,d]} \in \mathcal{R}[b,d]$. Then, applying the Proposition on the last integral, $\boxed{f|_{[b,c]} \in \mathcal{R}[b,c]}$ and $f|_{[c,d]} \in \mathcal{R}[c,d]$. □

Linearity is arguably the most important aspect of the definite integral.

Proposition 6.14. Let $f, g \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$. Then,

- $(f+g) \in \mathcal{R}[a,b]$, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$;
- $(cf) \in \mathcal{R}[a,b]$, and $\int_a^b (cf) = c\int_a^b f$.

A useful fact: $\inf(f+g)(x) \ge \inf f(x) + \inf g(x)$ and $\sup(f+g)(x) \le \sup f(x) + \sup g(x)$, where the supremum and the infimum are taken over $x \in S$ for some S with $f, g\colon S \to \mathbb{R}$. Also, the supremum and the infimum preserves non-strict inequalities for functions over a common domain.

Proof. We begin with the first item. For an arbitrary $P \in \mathcal{P}[a,b]$,

$$L(P, f+g) = \sum_{i=1}^n \inf(f+g)([x_{i-1}, x_i]) \cdot \Delta x_i \ge \sum_{i=1}^n (\inf f([x_{i-1}, x_i]) + \inf g([x_{i-1}, x_i]))) \cdot \Delta x_i = L(P, f) + L(P, g),$$

so taking the supremum of yields $\underline{\int_a^b}(f+g) \ge \underline{\int_a^b} f + \underline{\int_a^b} g = \int_a^b f + \int_a^b g$. Similarly,

$$U(P, f+g) = \sum_{i=1}^n \sup(f+g)([x_{i-1}, x_i]) \cdot \Delta x_i \le \sum_{i=1}^n (\sup f([x_{i-1}, x_i]) + \sup g([x_{i-1}, x_i])) \cdot \Delta x_i = U(P, f) + U(P, g).$$

Taking the infimum gives $\overline{\int_a^b}(f+g) \le \overline{\int_a^b} f + \overline{\int_a^b} g = \int_a^b f + \int_a^b g$. Thus,

$$\int_a^b f + \int_a^b g \le \underline{\int_a^b}(f+g) \le \overline{\int_a^b}(f+g) \le \int_a^b f + \int_a^b g.$$

Therefore, the non-strict inequalities must be equalities. Hence $(f+g) \in \mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

For the second item, first consider the case $c > 0$. We have by definition

$$\begin{cases} \underline{\int_a^b}(cf) = \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^n \inf(cf)([x_{i-1}, x_i]) \cdot \Delta x_i = c \sum_{i=1}^n \inf f([x_{i-1}, x_i]) \cdot \Delta x_i = c \underline{\int_a^b} f = \int_a^b f,^{10}, \\ \overline{\int_a^b}(cf) = \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^n \sup(cf)([x_{i-1}, x_i]) \cdot \Delta x_i = c \sum_{i=1}^n \sup f([x_{i-1}, x_i]) \cdot \Delta x_i = c \overline{\int_a^b} f = \int_a^b f. \end{cases}$$

We similarly conclude that the lower and upper Darboux sums are equals. Hence, $(cf) \in \mathcal{R}[a,b]$, and

$$\int_a^b (cf) = c \int_a^b f.$$

If $c < 0$, the same argument above applies AS-IS, with the portion in red and the portion in right interchanged in matching locations. If $c = 0$, then $(cf) \in \mathcal{R}[a,b]$ trivially. The proof is complete. □

Now monotonicity.

Proposition 6.15. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are bounded such that $\forall x \in [a,b], f(x) \le g(x)$. Then,

$$\underline{\int_a^b} f \le \underline{\int_a^b} g \quad \text{and} \quad \overline{\int_a^b} f \le \overline{\int_a^b} f.$$

Consequently, if $f, g \in \mathcal{R}[a,b]$, then

$$\int_a^b f \le \int_a^b g.$$

This is simply a corollary of the fact that $f(x) \le g(x)$ implies $\inf f(x) \le \inf g(x)$ and $\sup f(x) \le \sup g(x)$.

The big lemma: continuous functions on closed intervals are integrable.

Lemma 6.16. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Then, $f \in \mathcal{R}[a,b]$.

Proof. By the Heine-Borel theorem (Theorem 4.18), f is uniformly continuous. Suppose $\epsilon > 0$ is given. Fix $\delta > 0$ such that whenever $|x - y| < \delta$ $(x, y \in [a,b])$, $|f(x) - f(y)| < \epsilon/(b-a)$.

Choose an arbitrary $n \in \mathbb{N}$ such that $(b-a)/n < \delta$ and set $P = \{x_0, \cdots, x_n\}$, where $x_i = a + (b-a)i/n$ $(i = 0, \cdots, n)$. Then, for all $x, y \in [x_{i-1}, x_i]$, $|x - y| \le \Delta x_i = (b-a)/n < \delta$, hence $|f(x) - f(y)| < \epsilon/(b-a)$. Therefore, $\sup f([x_{i-1}, x_i]) - \inf f([x_{i-1}, x_i]) < \epsilon/(b-a)$. The equality is strict as a result of the Min-Max theorem (Theorem 4.15).

Hence,

$$\overline{\int_a^b} f - \underline{\int_a^b} f \le U(P,f) - L(P,f)$$

[10] $\inf cS = c \inf S$ for a positive c: For any $\epsilon > 0$ fix $x \in S$ such that $x - \inf S < \epsilon/c$, so $cx - c\inf S < \epsilon$, and hence $c \inf S = \inf\{cx\} = \inf cS$.

$$= \sum_{i=1}^{n} (\sup f([x_{i-1}, x_i]) - \inf f([x_{i-1}, x_i])) \cdot \Delta x_i$$

$$< \sum_{i=1}^{n} \frac{\epsilon}{b-a} \cdot \Delta x_i$$

$$= \epsilon.$$

Since the choice of $\epsilon > 0$ was arbitrary, we conclude that $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le 0$, and hence $f \in \mathcal{R}[a,b]$. □

A weirdly specific but somewhat useful result is presented below.

Lemma 6.17. Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences such that **(i)** $a < a_n < b_n < b$ for all $n \in \mathbb{N}$, **(ii)** $\lim_{n\to\infty} a_n = a$, **(iii)** $\lim_{n\to\infty} b_n = b$, and **(iv)** $f|_{[a_n,b_n]} \in \mathcal{R}[a_n, b_n]$ for all $n \in \mathbb{N}$. Then, $f \in \mathcal{R}[a,b]$, and

$$\int_a^b f = \lim_{n\to\infty} \int_{a_n}^{b_n} f.$$

Proof. Choose $M > 0$ such that $f([a,b]) \subset [-M, M]$. Then,

$$-M \cdot (b-a) \le -M \cdot (b_n - a_n) \le \int_{a_n}^{b_n} f \le M \cdot (b_n - a_n) \le M \cdot (b-a).$$

Therefore, the sequence $(\int_{a_n}^{b_n})_{n=1}^{\infty}$ is bounded. Let $(\int_{a_{n_k}}^{b_{n_k}} f)_{k=1}^{\infty}$ be an arbitrary convergent subsequence, whose existence is guaranteed by the Bolzano-Weierstrass theorem (Theorem 3.17). Thus, by Propositions 6.7 and 6.11, we have

$$\underline{\int_a^b} f = \underline{\int_a^{a_{n_k}}} f + \int_{a_{n_k}}^{b_{n_k}} f + \underline{\int_{b_{n_k}}^{b}} f \ge -M \cdot (a_{n_k} - a) + \int_{a_{n_k}}^{b_{n_k}} -M \cdot (b - b_{n_k}).$$

Taking the limit of the RHS as $k \to \infty$, we have

$$\underline{\int_a^b} f \ge -M \cdot 0 + \lim_{k\to\infty} \int_{a_{n_k}}^{a_{n_k}} f - M \cdot 0 = \lim_{k\to\infty} \int_{a_{n_k}}^{a_{n_k}} f.$$

Similarly, we have

$$\overline{\int_a^b} f \le \lim_{k\to\infty} \int_{a_{n_k}}^{a_{n_k}} f.$$

Therefore, $\underline{\int_a^b} f = \overline{\int_a^b} f = \lim \int_{a_{n_k}}^{a_{n_k}} f$.

We have concluded that every convergent subsequence of $(\int_{a_n}^{b_n} f)$ converges to a common value. Suppose now for the sake of contradiction that $(\int_{a_n}^{b_n} f)$ diverges. Then, since the sequence is bounded, Proposition 3.29 implies the limit superior does not equal the limit inferior. We can thus fix two subsequences converging to the limit superior and the limit inferior respectively by Proposition 3.28, which is a contradiction. Thus, $(\int_{a_n}^{b_n} f)$ converges, and by Proposition 3.15, we conclude that

$$\int_a^b f = \lim_{n\to\infty} \int_{a_n}^{b_n} f.$$

The proof is finished. □

We can now remove the restriction of continuity on the endpoints.

Proposition 6.18. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are bounded such that $f \in \mathcal{R}[a,b]$ and $f|_{(a,b)} = g|_{(a,b)}$. Then, $g \in \mathcal{R}[a,b]$, and

$$\int_a^b f = \int_a^b g.$$

Proof. This is an immediate conclusion of Lemma 6.17. Define two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ in (a, b) as

$$\begin{cases} a_n = a + \dfrac{b-a}{3n}, \\ a_n = b - \dfrac{b-a}{3n}. \end{cases}$$

Clearly, $a < a_n < b_n < b$ for all $n \in \mathbb{N}$, $\lim a_n = a$, and $\lim b_n = b$. Further, since $f|_{(a,b)} = g|_{(a,b)}$, we have $g|_{[a_n,b_n]} = f|_{[a_n,b_n]} \in \mathcal{R}[a, b]$ as a result of Corollary 6.13. Thus, $g \in \mathcal{R}[a, b]$, and

$$\int_a^b g = \lim_{n\to\infty} \int_{a_n}^{b_n} f = \lim_{n\to\infty} \int_{a_n}^{b_n} g = \int_a^b f.$$

The proof is completed. □

It is easy to now show that a bounded function with finitely many discontinuities is necessarily integrable. Also, if $f \in \mathcal{R}[a, b]$ and $g\colon [a, b] \to \mathbb{R}$ is bounded, where f and g only differ on a finite subset of the domain, then $g \in \mathcal{R}[a, b]$ and $\int_a^b f = \int_a^b g$.

References

[1] Jiří Lebl. Basic analysis: Introduction to real analysis. 2009.

A Dedekind Cuts

Definition A.1. A subset of rational numbers $A \subset \mathbb{Q}$ is said to be a Dedekind[11] cut of $\mathbb{Q}$ iff it satisfies $A \neq \emptyset$ and $A \neq \mathbb{Q}$ with the following two properties:

- $\forall x \in A, \forall x' \in \mathbb{Q}, x' < x \Rightarrow x' \in A$; consequently, A is not bounded from below;
- $\forall x \in A, \exists x' \in A, x < x'$; that is, A does not have a largest element.

The set of all Dedekind cuts of $\mathbb{Q}$ is denoted as $\mathcal{D} \subset 2^{\mathbb{Q}}$.

The key idea here is that we are manually constructing $\{(-\infty, x)_{\mathbb{Q}} \mid x \in \mathbb{R}\}$, which $\mathcal{D}$ is isomorphic to. Of course, the notation isn't justified until we have finished constucting $\mathbb{R}$.

The first thing to do here is to give a good definition of "equality." Are there technically different (i.e., not equal as sets) $A, B \in \mathcal{D}$ that should be considered the same number? We saw this before when we defined the set of integers $\mathbb{Z}$ as *equivalence classes* on $\mathbb{Z}^2_{\geq 0}$. We said that two integers $(a, b), (c, d) \in \mathbb{Z}^2_{\geq 0}$ are said to be equal iff

$$a + d = b + c,$$

because the integer represented by (a, b) is really "$a - b$." In this sense, we have "more" numbers in $\mathbb{Z}^2_{\geq 0}$ than we need in $\mathbb{Z}$[12] So, $\mathbb{Z}$ is *not* $\mathbb{Z}^2_{\geq 0}$ *per se* but that product up to the equivalence relation described above. However, we'll see that this is not an issue for us here.

Since our construction of $\mathbb{R}$ isn't as "explicit" as, say, $\mathbb{Z}$, we need to characterize the behavior of this equivalence relation we want in a somewhat roundabout way. We'll first define a partial ordering, and then show that for any two Dedekind cuts A and B, either $A \leq B$ or $B \leq A$, which will suffice for us to show that each set represents a unique real number. In this sense, $\mathcal{D}$ *is* just $\mathbb{R}$; there are no duplicates of representations of real numbers in $\mathcal{D}$.

Definition A.2. The partial ordering $\leq$ is defined on $\mathcal{D}$ as follows. Let $A, B \in \mathcal{D}$. We say that $A \leq B$ iff $A \subseteq B$.

It is clearly reflexive, antisymmetric, and transitive. Let's now show that indeed any two cuts $A, B \in \mathcal{D}$ can be compared; that is,

$$\forall A, B \in \mathcal{D}, A \leq B \vee B \leq A.$$

Proof. Let A, B be Dedekind cuts of $\mathbb{Q}$.

If there exists x with $x \in A$ but $x \notin B$, then $S := \{x' \in \mathbb{Q} \mid x' \leq x\} \subseteq A$ by definition. Since $x \notin B$, so $x' \leq x$ for any $x' \in B$. Further, by definition, we can choose some $\tilde{x} \in A$ strictly greater than x, so

$$\forall x' \in B, x' \leq x < \tilde{x}.$$

Since $\tilde{x} \in A$, we conclude that $\forall x' \in B, x' \in A$. Thus, $B \subseteq A$, so $B \leq A$.

If there exists x with $x \in B$ but $x \notin A$, we may apply the same argument to obtain $A \subseteq B$, or $A \leq B$.

Otherwise, we have $A \backslash B = \emptyset$ or $B \backslash A = \emptyset$. This implies that $A \subseteq B$ or $B \subseteq A$, which still gives $A \leq B$ or $B \leq A$. The proof is completed. ◻

Sweet. In our proof, we repeatedly leverage the two properties of Dedekind cuts, both of which are characterizations of the open interval $(-\infty, x)_{\mathbb{Q}}$ for $x \in \mathbb{R}$. This also reassures us that we don't need to define another equivalence relation, different from the "=" relation, to characterize the equality of real numbers.

[11] This is pronounced as /ˈdeiːdəˌkɪnt/, similar to DAY-duh-kint in English.
[12] Of course they're still the same in cardinality.

Alright. Let's first say that $\forall x \in \mathbb{Q}, (-\infty, x)_{\mathbb{Q}} \in \mathcal{D}$. Note that here, $x \in \mathbb{Q}$, so we aren't dabbling with self-referencing or other weird shii like that. We will denote $(-\infty, x)_{\mathbb{Q}}$ with $[x]$.[13] We'll also prove that the two partial orderings defined match; that is, $[x] \leq [y] \Leftrightarrow x \leq y$ for $x, y \in \mathbb{Q}$.

Proof. Fix an arbitrary $x \in \mathbb{Q}$. Then, $[x] = \{x' \in \mathbb{Q} \mid x' < x\}$. Clearly, $[x] \neq \varnothing$ and $[x] \neq \mathbb{Q}$. We now verify the two properties of Dedekind cuts.

Fix $u \in [x]$ and $v \in \mathbb{Q}$ with $v < u$. Then, $v < u < x$, so $v \in (-\infty, x)_{\mathbb{Q}} = [x]$. The first property is satisfied.

Fix $u \in [x]$. Let $v = (u + x)/2$, so $u < v < x$ and thus $v \in [x]$. The second property is also satisfied. We conclude that $[x]$ is indeed a Dedekind cut.

Now suppose $x, y \in \mathbb{Q}$. If $x = y$, then $[x] = [y]$. If $x < y$, then $[x] \subsetneq [y]$, so $[x] < [y]$. If $x > y$, then for the same reason we have $[x] > [y]$. The proof is finished. □

In addition, we can safely claim that $[x]$ is unique for any $x \in \mathbb{Q}$, so we've successfully captured all the rational numbers in $\mathcal{D}$, with no duplicates. We say that $A \in \mathcal{D}$ is rational iff $\exists x \in \mathbb{Q}, [x] = A$.

Proof. Suppose that $[x] = [y]$ for $x, y \in \mathbb{Q}$. Then, $(-\infty, x)_{\mathbb{Q}} = (-\infty, y)_{\mathbb{Q}}$. Suppose for the sake of contradiction that $x \neq y$. Without loss of generality, assume $x < y$. Then, $x < (x + y)/2 < y$, so $(x + y)/2 \notin [x]$ but $(x + y)/2 \in [y]$, so $[x] \neq [y]$, which is a contradiction. The proof is finished. □

The next step is to define addition and the additive inverse as well as multiplication and the multiplicative inverse.

Addition is super straightforward so I'll just state it here.

Definition A.3. Let $A, B \in \mathcal{D}$. We define $A + B \coloneqq \{a + b \mid a \in A, b \in B\}$.

This is obviously a Dedekind cut and further, compatible with rational addition. We will prove the latter statement.

Proof. Observe that $[x] + [y] \triangleq \{u + v \mid u \in [x], v \in [y]\} = \{u + v \mid u, v \in \mathbb{Q} \land u < x \land v < y\} = \{(u + v) \in \mathbb{Q} \mid (u + v) < x + y\} = [x + y]$, which completes the proof. □

And it's compatible with field axioms (A1) through (A4), stated without proof just because it's super straightforward and I'm way too lazy to do that.

Additive inverse is a weird thing... I guess we can flip it and then take the complement. But then if it's representing a rational number x, we need to also take out x from $-x \triangleq (-\infty, -x]_{\mathbb{Q}}$ to get $(-\infty, -x)_{\mathbb{Q}} = [-x]$.

Definition A.4. Let $A \in \mathcal{D}$. If $[a] = A$ for some $a \in \mathbb{Q}$, then we define $-A \coloneqq (-\infty, -a)_{\mathbb{Q}}$. Otherwise, we define $-A \coloneqq \{-a \mid a \in \mathbb{Q} \backslash A\}$.

Since the construction isn't as natural, we should show that it is closed in $\mathcal{D}$. In order to do that, let's first establish another property of Dedekind cuts of $\mathbb{Q}$.

Proposition A.5. Let $A \in \mathcal{D}$ be a Dedekind cut of $\mathbb{Q}$. For any $b \in \mathbb{Q}$, b is an upper bound of A in $\mathbb{Q}$ if and only if $b \notin A$.

Proof. We first prove that if b is an upper bound of A in $\mathbb{Q}$, then $b \notin A$. Suppose for the sake of contradiction that $b \in A$ is an upper bound of A. Since $b \in A$, by definition, we may choose some $b' \in A$ such that $b' > b$. However, this contradicts the fact that b is an upper bound.

Let us now show that if $b \notin A$, then b is an upper bound. Negating property 1 of Dedekind cuts, we have $\forall x \in A, b \geq x$, so b is indeed an upper bound. The proof is finished. □

[13] Not to be confused with the floor function $\lfloor \cdot \rfloor$.

Now that we have all we need, let's start.

Proof. Let $A = [x]$, where $x \in \mathbb{Q}$. Then, $[-x] = (-\infty, -a)_{\mathbb{Q}}$, which we have shown must be in $\mathcal{D}$.

Otherwise, we have $-A = \{-a \mid a \in \mathbb{Q}\backslash A\}$. Clearly, $-A \neq \varnothing$ and $-A \neq \mathbb{Q}$. We examine the first property.

Choose an arbitrary $x \in -A$ and $x' \in \mathbb{Q}$ with $x' < x$. Since $x \in -A$, we have $-x \notin A$, so $\forall a \in A, -x \geq a$. Since $x' < x$, by transitivity, we have $\forall a \in A, -x' \geq a$, so $-x'$ is an upper bound of A. By Prop. A.5, we conclude that $-x' \notin A$, or $-x' \in \mathbb{Q}\backslash A$. Therefore, $x' \in -A$. The first property is satisfied.

Now fix an arbitrary $x \in -A$, so $-x \in \mathbb{Q}\backslash A$. Then, by Prop. A.5, we conclude that $b := -x$ is an upper bound of A and is not in A. Suppose for the sake of contradiction that $\forall b' \in \mathbb{Q}, b' < b \Rightarrow b' \in A$. By supposition, we can already conclude that $[b] \subseteq A$. Now fix an arbitrary $a \in A$. Since b is an upper bound of A, we have $b \geq a$. Since $b \notin A$, it is impossible that $a = b$; thus, $a < b$. Therefore, $\forall a \in A, a < b$. Therefore, $A \subseteq [b]$, and we conclude that $A = [b]$. This is a contradiction, since $A \neq [x]$ for any $x \in \mathbb{Q}$. The proof is completed. □

Well, weird definition, but at least it's defined to be compatible with rational additive inverses. We'll need to show that it's also compatible with field axiom (A5); that is, $\forall A \in \mathcal{D}, A + (-A) = [0]$. Before we dive in, let's prove another property of Dedekind cuts that captures the fact that it has some sort of "least upper bound," even though it may not be a rational number *per se.*

Proposition A.6. Suppose $A \in \mathcal{D}$. Then $\forall \epsilon > 0, \exists a \in A, a + \epsilon \notin A$.

Proof. Suppose on the contrary that there exists some $\epsilon > 0$ such that $\forall a \in A, a + \epsilon \in A$. Induction allows us to conclude that, in fact, $\forall n \in \mathbb{N}, \forall a \in A, a + n\epsilon \in A$. Choose an arbitrary $x \in \mathbb{Q}$. Since $\epsilon > 0$, we may use the Archimedean property of rationals to justify our choice of some $n \in \mathbb{N}$ such that $n\epsilon > x - a \in \mathbb{Q}$. Then, $a + n\epsilon > x$. Since $a + n\epsilon \in A$, we conclude by definition that $x \in A$. Since the choice of x was arbitrary, we have $\mathbb{Q} \subseteq A$, which contradicts the fact that $A \in \mathcal{D} \subset 2^{\mathbb{Q}}$ and $A \neq \mathbb{Q}$. The proof is finished. □

I want to point out that it doesn't matter whether $\epsilon \in \mathbb{R}_+$ or $\epsilon \in \mathbb{Q}_+$; the effects are equivalent at least throughout this article.

Now let's prove that indeed our definition of the additive inverse is compatible with (A5).

Proof. Suppose $A \in \mathcal{D}$. If $[a] = A$ for some $a \in \mathbb{Q}$, then $[a] + [-a] = \{x \in \mathbb{Q} \mid x < a + (-a) = 0\} = [0]$.

Otherwise, we have

$$A + (-A) = A + \{-b \mid b \in \mathbb{Q}\backslash A\} = \{a - b \mid a \in A, b \in Q\backslash A\}.$$

In order for this to be equal to $[0]$, we need to show both $A + (-A) \subseteq [0]$ and $[0] \subseteq A + (-A)$.

Suppose $a \in A$ and $b \in \mathbb{Q}\backslash A$. Then, by construction, $b \notin A$, so $b \geq a$ by definition. Suppose now for the sake of contradiction that $a = b$. Then, $a = b \in A \cap (\mathbb{Q}\backslash A) = \varnothing$, which is impossible. Thus, we can conclude that $b > a$, and thus $a - b < 0$. That is, $A + (-A) \subseteq [0]$.

Now we need to show that $[0] \subseteq \{a - b \mid a \in A, b \in \mathbb{Q}\backslash A\}$, or equivalently,

$$\forall x \in \mathbb{Q}, x < 0 \Rightarrow \exists a \in A, \exists b \in \mathbb{Q}\backslash A, a - b = x.$$

Suppose $x < 0$ is rational. Let $\epsilon = -x > 0$. Applying Prop. A.6, we conclude that there exists some $a \in A$ such that $a + \epsilon = a - x \notin A$. Let $b = a - x$, then $b \in \mathbb{Q}\backslash A$. Then, we have $a - b = x$, and the proof is completed. □

Multiplication is a bit trickier, since we need to consider the sign. For this reason, I'll be verbose and give the definition of positive reals, negative reals, and absolute values.

Definition A.7. Let $A \in \mathcal{D}$ be a Dedekind cut of $\mathbb{Q}$. A is said to be positive if $A > [0]$ and negative if $A < [0]$. We define

$$|A| = \begin{cases} A, & A \geq [0], \\ -A, & \text{otherwise.} \end{cases}$$

Now let's define multiplication:

Definition A.8. Let $A, B \in \mathcal{D}$ be Dedekind cuts of $\mathbb{Q}$. Suppose $A \leq [0]$ and $B \leq [0]$. Define

$$\Pi \coloneqq \{ab \mid a \in A, b \in B\}.$$

Then, the product of A and B is defined as

$$A \cdot B \coloneqq \{x \in \mathbb{Q} \mid \exists \epsilon > 0, x + \epsilon \notin \Pi\}.$$

If exactly one of the two is positive, then the product is defined as $A \cdot B \coloneqq -(-|A|) \cdot (-|B|)$. If both are positive, then the product is defined as $A \cdot B \coloneqq (-A) \cdot (-B)$.

Since the construction isn't as natural, we should show that multiplication is closed in $\mathcal{D}$.

Proof. Let $A, B \in \mathcal{D}$ be a Dedekind cut of $\mathbb{Q}$.

If $A \leq [0]$ and $B \leq [0]$, then define $\Pi \coloneqq \{ab \mid a \in A, b \in B\}$, so $A \cdot B = \{x \in \mathbb{Q} \mid \exists \epsilon > 0, x + \epsilon \notin \Pi\}$. Clearly, $\Pi \neq \varnothing$ and $\Pi \neq \mathbb{Q}$ by construction (why?), so $A \cdot B$ is neither empty nor equal to $\mathbb{Q}$.

Let us first prove an important proposition that $\forall \alpha \in \Pi, \forall \beta \in \mathbb{Q}, \beta > \alpha \implies \beta \in \Pi$. Let $\alpha = ab$ where $a \in A$ and $b \in B$. Then,

$$\beta = \alpha + \beta - \alpha = ab + (\beta - \alpha) \cdot a/a = a \cdot \left(b + \frac{\beta - \alpha}{a}\right).$$

Clearly, $a < 0$. Since $\beta > \alpha$, we have $\beta - \alpha > 0$ and thus $\dfrac{\beta - \alpha}{a} < 0$. Therefore, $b + \dfrac{\beta - \alpha}{a} < b$, and from property 1 of Dedekind cuts we conclude that $b' \coloneqq \left(b + \dfrac{\beta - \alpha}{a}\right) \in B$. Since $\beta = ab'$ for some $a \in A$ and $b' \in B$, we conclude that $\beta \in \Pi$.

Now we show that the two properties of Dedekind cuts are satisfied. Choose arbitrary $x \in A \cdot B$ and $x' \in \mathbb{Q}$ such that $x' < x$. Suppose for the sake of contradiction that $x' \notin A \cdot B$, so $x' + \epsilon \in \Pi$ for any $\epsilon > 0$. Note that $x' < x$, so $x' + \epsilon < x + \epsilon$. From the proposition proven above (with $\alpha = x' + \epsilon$ and $\beta = x + \epsilon$), we conclude that $x + \epsilon \in \Pi$, which contradicts the fact that $x \in A \cdot B$. The first property has been proven.

Fix arbitrary $x \in A \cdot B$. Then, there exists some $\epsilon_0 > 0$ such that $x + \epsilon_0 \notin \Pi$. Now consider $x' \coloneqq x + \epsilon_0/3$. Then, if $\epsilon = \epsilon_0/3$, we have $x' + \epsilon = x + 2/3 \cdot \epsilon_0 < x + \epsilon_0$. Suppose $x' + \epsilon \in \Pi$. Then, since $x + \epsilon_0 > x' + \epsilon$, $x + \epsilon_0$ must be an element of Π, which is a contradiction. Thus, $x' + \epsilon \notin \Pi$ for some $\epsilon > 0$. Therefore, $x' \in A \cdot B$. The second property has been shown.

If exactly one of A and B is non-positive, then by definition $A \cdot B = -|(-|A|) \cdot (-|B|)|$. Since $-|A| \leq [0]$ and $-|B| \leq [0]$, we conclude that $(-|A|) \cdot (-|B|) \in \mathcal{D}$, and thus $A \cdot = -|(-|A|) \cdot (-|B|)| \in \mathcal{D}$ since the additive inverse is closed in $\mathcal{D}$.

If both are positive, then for similar reasons, we conclude $A \cdot B \in \mathcal{D}$. The proof is completed. □

Let's show that this is compatible with rational multiplication.

Proof. Suppose $x, y \in \mathbb{Q}$ are both non-positive. Let $\Pi = \{ab \mid a \in [x], b \in [y]\}$. Then, $\Pi = \{ab \mid a < x < 0, b < y < 0\} = (xy, +\infty)_{\mathbb{Q}}$. Then, $[x][y] = \{t \in \mathbb{Q} \mid \exists \epsilon > 0, t + \epsilon \notin \Pi\} = \{t \in \mathbb{Q} \mid \exists \epsilon > 0, t + \epsilon \leq xy\}$. If $t < xy$, then let $\epsilon = (xy - t)/2$, and we have $t + \epsilon < xy \leq xy$. If $t \geq xy$, then for any $\epsilon > 0$ we have $t + \epsilon \geq xy + \epsilon > xy$, so $t \notin [x][y]$. Thus, we conclude that $[x][y] = [xy]$.

Suppose exactly one of the two is positive. Without loss of generality, assume that $x \leq 0 < y$. Then, $[x][y] = -[-x][y] = -[-xy] = [xy]$.

Lastly, if both are positive, then $[x][y] = [-x][-y] = [xy]$. □

This is also clearly compatible with field axioms (M1) through (M4), stated without proof.

Now it is time to establish the multiplicative inverse.

Definition A.9. Let $A \in \mathcal{D}$ be a non-zero Dedekind cut of $\mathbb{Q}$. If A is negative, then define

$$\tilde{A} := \{1/a \mid a \in A\}.$$

Its multiplicative inverse is then defined as

$$A^{-1} := \{x \in \mathbb{Q}_- \mid \exists x' \in \mathbb{Q}_-, x' > x \land x' \notin \tilde{A}\}.$$

If A is positive, then its multiplicative inverse is defined as $-(-A)^{-1}$.

Let's first show that A^{-1} is indeed a Dedekind cut.

Proof. Let $A \in \mathcal{D}$ be a non-zero Dedekind cut of $\mathbb{Q}$.

Suppose $A < [0]$. Define $\tilde{A} := \{1/a \mid a \in A\}$. Then, $A^{-1} = \{x \in \mathbb{Q}_- \mid \exists x' \in \mathbb{Q}_-, x' > x \land x' \notin \tilde{A}\}$. Clearly, $A^{-1} \neq \varnothing$ and $A^{-1} \neq \mathbb{Q}$.

We now demonstrate the first property of Dedekind cuts is satisfied. Choose arbitrary $x \in A^{-1}$ and $x' \in \mathbb{Q}$ such that $x' < x$. Fix $\tilde{x} \in \mathbb{Q}_-$ such that $\tilde{x} > x$ and $\tilde{x} \notin \tilde{A}$. Since $x' < x < \tilde{x}$ and $\tilde{x} \notin \tilde{A}$, we conclude that $x' \in A^{-1}$ by definition.

Now we demonstrate the second property. Choose arbitrary $x \in A^{-1}$. Fix $\tilde{x} \in \mathbb{Q}_-$ such that $\tilde{x} > x$ and $\tilde{x} \notin \tilde{A}$. Let $x' = (x + \tilde{x})/2$. Then, $x < x' < \tilde{x}$. Since $\tilde{x} \notin \tilde{A}$, we conclude that $x' \in A^{-1}$ by definition.

If $A > [0]$, then A^{-1} is by definition $-(-A)^{-1}$. $-A \in \mathcal{D}$ since the additive inverse is closed. Then, $(-A)^{-1} \in \mathcal{D}$ since the reciprocal has been shown to be closed for negative Dedekind cuts. Applying the closedness of additive inverses again, we conclude that $A^{-1} = -(-A)^{-1} \in \mathcal{D}$. The proof is finished. □

Now let's prove that this is compatible with rational reciprocals.

Proof. Suppose $x \in \mathbb{Q}$ is negative. Define $\tilde{X} := \{1/a \mid a < x < 0\} = (1/x, 0)_{\mathbb{Q}}$. Then,

$$\begin{aligned}
[x]^{-1} &= \{t \in \mathbb{Q}_- \mid \exists t' \in \mathbb{Q}_-, t' > t \land t' \notin \tilde{X}\} \\
&= \{t \in \mathbb{Q}_- \mid \exists t' \in \mathbb{Q}_-, t' > x \land t' \leq 1/x\} \\
&= \{t \in \mathbb{Q}_- \mid t < 1/x\} \\
&= (-\infty, 1/x)_{\mathbb{Q}} \\
&= [1/x].
\end{aligned}$$

Now suppose $x \in \mathbb{Q}$ is positive. Then, $[x]^{-1} = -[-x]^{-1} = -[1/(-x)] = -[-1/x] = [1/x]$. The proof is finished. □

And now we show that it satisfies axiom (M5); that is $A \cdot A^{-1} = [1]$ for any non-zerp $A \in \mathcal{D}$. This is quite tricky: to find $A \cdot A^{-1}$ we need to expand A^{-1}, which involves $\tilde{A}$, which is dependent on A. We have to cross this long bridge of logical connectives to show that this is indeed the set $[1]$.

Proof. Let $A \in \mathcal{D}$ be non-zero.

Suppose $A < [0]$. Then, let $\tilde{A} := \{1/a \mid a \in A\}$. So, $A^{-1} = \{x \in \mathbb{Q}_- \mid \exists x' \in \mathbb{Q}_-, x' > x \wedge x' \notin \tilde{A}\}$. In order to show that $A \cdot A^{-1} = [1]$, it suffices to show that $\Pi := \{ab \mid a \in A, b \in A^{-1}\}$ is equal to $(1, +\infty)_{\mathbb{Q}}$.

Choose arbitrary $a \in A$ and $b \in A^{-1}$. Then, there exists some $b' \in \mathbb{Q}_-$ with $b' > b$ such that $b' \notin \tilde{A}$. We first show that b' must be a lower bound of $\tilde{A}$. Since $b' \notin \tilde{A}$, we have $\forall a' \in A, b' \neq 1/a'$. Suppose for the sake of contradiction that $b' > 1/a'$. Then, $a' > 1/b'$, and thus $1/b' \in A$ by the first property of Dedekind cuts. Consequently, $b' \in \tilde{A}$ by definition, which is a contradiction. Therefore, $b < b' < 1/a'$ for any $a' \in A$; that is, b and b' are both strict lower bounds of $\tilde{A}$. Since $1/a \in \tilde{A}$, we have $b < 1/a$. Since $a \in A < [0]$, $a < 0$, and thus $ab > 1$. Therefore, we conclude that $\Pi \subseteq (1, +\infty)_{\mathbb{Q}}$.

Now we show that $x \notin \Pi$ for any rational $x \leq 1$. Since $ab > 1 \geq x$ for any $a \in A$, $b \in A^{-1}$, we have $ab > x$ and thus $ab \neq x$. Therefore, $x \notin \Pi$. We conclude that $\Pi = (1, +\infty)_{\mathbb{Q}}$.

By definition, $A \cdot B = \{t \in \mathbb{Q} \mid \exists \epsilon > 0, t + \epsilon \notin \Pi\}$. Suppose $t \in \mathbb{Q}$. We aim to show that $t < 1$ if and only if $\exists \epsilon > 0, t + \epsilon \leq 1$. If $t < 1$, then let $\epsilon = (1 - t)/2$, so $t + \epsilon = (t + 1)/2 \leq 1$. If $t \geq 1$, then for any $\epsilon > 0$ we have $t + \epsilon > t \geq 1$, so $t + \epsilon > 1$. Therefore, we conclude that $A \cdot B = [1]$. By similar reasoning, if $A > [0]$ and $B := A^{-1}$, $A \cdot B = [1]$. The proof is finished. □

Now we have obtained both the ordering on $\mathcal{D}$ and the four arithmetic operations on $\mathcal{D}$, which makes $\mathcal{D}$ an ordered field.

Theorem A.10. *The set of all Dedekind cuts in $\mathbb{Q}$, $\mathcal{D}$, is an ordered field.*

Before we introduce powers, we should first prove that it is *complete*. There are many equivalent formulations of completeness:

- Any bounded subset of $\mathcal{D}$ has a supremum;
- Any Cauchy sequence in $\mathcal{D}$ converges in $\mathcal{D}$;
- There is some element of $\mathcal{D}$ contained in any term of any sequence of nested intervals in $\mathcal{D}$.

Of particular interest to us are the first two. We will demonstrate the first here.

Theorem A.11. *Let $\mathcal{D}$ be the set of all Dedekind cuts in $\mathbb{Q}$. Suppose $S \subset \mathcal{D}$ is bounded from above. Then it admits the least upper bound.*

Proof. Consider the union of all the Dedekind cuts in S:

$$\tilde{S} := \bigcup_{A \in S} A.$$

We claim that $\tilde{S}$ is a Dedekind cut. Clearly, $\tilde{S} \neq \varnothing$.

We first show that $\tilde{S} \neq \mathbb{Q}$. Since S is bounded from above, we may choose an arbitrary upper bound $B \in \mathcal{D}$. Then, $\forall A \in S, A \subseteq B$. Therefore, the union $\tilde{S} \subseteq B$. Since $B \neq \mathbb{Q}$, we conclude that $\tilde{S} \neq \mathbb{Q}$.

Now we demonstrate that the first property of Dedekind cuts are satisfied. Choose arbitrary $s \in \tilde{S}$ and $s' \in \mathbb{Q}$ such that $s' < s$. Then, there exists some $A \in S$ such that $s \in A$. Since $s' < s$, we conclude that $s' \in A$, so $s' \in \tilde{S}$.

We now show that the second property also holds. Choose arbitrary $s \in \tilde{S}$. Then, there exists some $A \in S$ such that $s \in A$. Applying the second property of Dedekind cuts to A, we may choose some $s' \in A$ such that $s' > s$. Since $s' \in A$, we have $s' \in \tilde{S}$. Therefore, there exists some $s' \in \tilde{S}$ such that $s' > s$ for any $s \in \tilde{S}$. We conclude that $\tilde{S}$ is indeed a Dedekind cut of $\mathbb{Q}$.

We now assert that $\tilde{S}$ is the least upper bound of S. First, $\tilde{S}$ is an upper bound since for any $A \in S$, $A \subseteq \bigcup_{A' \in S} A' = \tilde{S}$, so $A \leq \tilde{S}$. Further, suppose $B \in \mathcal{D}$ is also an upper bound of S; that is, $\forall A \in S, B \geq A$. So, $A \subseteq B$ for any $A \in S$, so $\tilde{S} = \bigcup_{A \in S} A \subseteq B$. Therefore, $\tilde{S} \leq B$ for any upper bound $B \in \mathcal{D}$ of S. The proof is finished. □

The construction of real numbers is now finished. We make a new set $\mathbb{R}$ that uniquely corresponds to elements in $\mathcal{D}$. For any $x \in \mathbb{Q}$, we let $x \in \mathbb{R}$ correspond to $[x] \in \mathcal{D}$. We inherit the ordering and the field operations from $\mathcal{D}$ and, of course, the completeness.

Before we finish, allow me to give an elegant proof of the Archimedean property of real numbers.

Theorem A.12. *For any $x, y \in \mathbb{R}$ with $x > 0$, there exists some $n \in \mathbb{N}$ such that $nx > y$.*

Proof. Suppose $x, y \in \mathbb{R}$ with $x > 0$. Suppose on the contrary that for any $n \in \mathbb{N}$, $nx \leq y$. Let $S := \{nx \mid n \in \mathbb{N}\}$. Then, y is an upper bound of S. By the least upper bound property of real numbers (Theorem A.11), there exists some $b = \sup S$. For any $n \in \mathbb{N}$, $(n + 1) \in \mathbb{N}$, so $(n + 1)x \leq b$. This is equivalent to $\forall n \in \mathbb{N}, nx \leq b - x$; that is, $(b - x)$ is also an upper bound of S. However, $b - x < b$ since $x > 0$, which contradicts the fact that b is the *least* upper bound. The proof is finished. □